Basic notions of abstract algebra

A **magma** is a set $\{g_1, g_2, ..\}$ together with a composition law $\circ$ such that $\forall g$

$$g_1 \circ g_2 = g_3 \in \text{set} \qquad \textit{closure}$$

A **semigroup** is a magma such that $\forall g$

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \qquad \textit{associativity}$$

A **monoid** is a semigroup such that $\forall g$

$$\exists\, e : e \circ g = g \circ e = g \qquad e \textit{ unit element}$$

A **group** G is a monoid such that $\forall g$

$$\exists\, g^{-1} : g^{-1} \circ g = g \circ g^{-1} = e \qquad g^{-1} \textit{ inverse element}$$

A **ring** R is set together with two composition laws $+$ and $\cdot$ (addition and multiplication respectively) such that the set is group under addition and is a monoid under multiplication[1]. The composition $+$ is abelian while $\cdot$ is not required to be abelian and $\cdot$ is distributive with respect to $+$.

Example: The set of integers $\mathbb{Z}$ is a ring. The number 0 is the identity under addition and the number 1 is the identity under multiplication. There is no notion of inverse element with respect to the multiplication.

A **field** K is a set together with two composition laws $+$ and $\cdot$ (addition and multiplication respectively) such the set is an abelian group under $+$ and is an abelian group under $\cdot$ for all the nonzero elements, and $\cdot$ is distributive with respect to $+$.

Example: The set of rational numbers $\mathbb{Q}$ is a field. Others fields are the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$.

A **linear vector space** V over a field K consists of two sets:
1 - a field K
2 - an abelian group V with elements $u, v...$ called vectors and a composition law[2] $+$: $V \times V \to V$ such that $\forall \alpha, \beta \in K$ and $\forall u, v \in V$ $\exists$ a map $K \cdot V \to V$ called the scalar multiplication such that

$$\begin{aligned} &\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u && \textit{associativity} \\ &\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v && \textit{distributivity} \\ &(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u && \textit{distributivity} \end{aligned}$$

The unit element with respect to $+$ in V is called the zero-vector.

Example: The set of arrows pointing in a direction $\vec{v}$ satisfy the conditions above and therefore are called vectors.

In the abstract sense, any element of a set which satisfies the conditions of a linear vector space is a vector: ordinary functions are vectors, matrices are

[1]Some authors define a ring as a set where the multiplication is a semigroup. In this case, if the identity element exists the ring is called a unital ring.

[2]this composition law $+$ in V is indicated with the same symbol of the composition in the field K. These two compositions are distinct.

vectors as well as tensors and linear connections.

A R-**module** satisfies the same conditions of a vector space but instead of being over a field K it is over a ring R.

An **algebra** is a vector space together with a composition law $\square$ which is a magma. Additional proprieties satisfied by $\square$ give different kinds of algebra.

$$\begin{array}{lll} \text{If} \quad \forall u \; \exists \; e \in V: & e\square u = u\square e = u & \textit{unital} \\ \text{If} \quad \forall u, v, w & u\square(v\square w) = (u\square v)\square w & \textit{associative} \\ \text{If} \quad \forall u, v & u\square v = v\square u & \textit{commutative} \end{array}$$

e, the unit of A, in general is different from the zero-vector.

Example: V = set of functions $f : \mathcal{M} \to \mathbb{R}$, $K = \mathbb{R}$ and $\square$ = ordinary multiplication of functions, $e = f(x) = 1$. All of the above proprieties are satisfied.

Example: V = set of the matrices A, $K = \mathbb{R}$ and $\square$ = matrix multiplication, $e = \mathbb{1}$. The algebra is unital and associative but not commutative.

Example: Lie Algebra. $V = \mathbb{R}^n$, $K = \mathbb{R}$ or $\mathbb{C}$ and $\square = [\ ,\ ]$ the Lie Bracket which satisfies:

$$\begin{array}{ll} [u, v] = -[v, u] & \textit{anticommutativity} \\ [\alpha u + \beta v, w] = \alpha[u, v] + \beta[v, w] & \textit{bilinearity} \\ [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 & \textit{Jacoby} \end{array}$$

Lie algebra are not unital, not associative and not cummutative.

Example: $V = \mathbb{R}^3$, $K = \mathbb{R}$, $\square$ = the cross product of ordinary vectors $\vec{v}$. It's a Lie algebra since the cross product satisfies the conditions of the Lie bracket. With e_i a basis of V^3, the cross product reads $e_i \times e_j = \epsilon_{ijk} e_k$ where ϵ_{ijk} is the Levi-Civita symbol. It is the algebra $SO(3)$. There is no unit since $\nexists\, \vec{e} : \vec{v} \times \vec{e} = \vec{v}$.

Example: Poisson algebra. V = set of smooth functions $f \in C^\infty(\mathcal{M})$, $K = \mathbb{R}$ or $\mathbb{C}$ and with two $\square$ compositions. The first $\square$ = ordinary multiplication of functions, which makes the Poisson algebra commutative and associative. The second $\square$ = Poisson bracket which is a Lie Bracket satisfying the additional condition:

$$[u, vw] = [u, v]w + v[u, w] \qquad \textit{Leibniz.}$$

Notice how the symbol vw only makes sense because the first composition is defined.