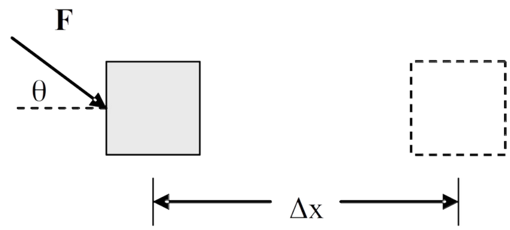
Work and Energy

The concept of energy is of fundamental importance in physics since everything which exists in Nature does have energy. Energy comes in many forms depending on the physical system considered. We start by introducing the *mechanical energy*. The mechanical energy is defined as the ability of an object to do *work* because of its state of rest at some particular location or motion.

Work

A constant force F is applied to an object resulting in the object moving over a displacement Δx. The *work* is done by the force is the dot product of the force times the displacement.

$$W = \vec{F} \cdot \Delta \vec{x}$$ [Nm = joule (J)]



If the force makes an angle with the displacement then

$$W = F\cos(\theta)\Delta x = F_x \Delta x$$

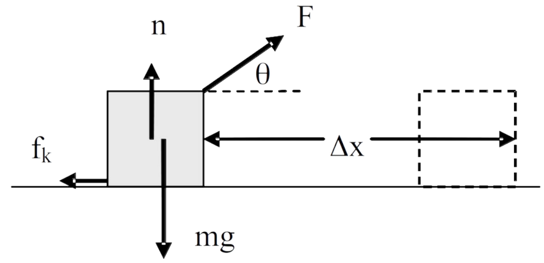
where θ is the angle between the force and the displacement and F_x is the x-component of the force.
If **F** is perpendicular to the displacement ($\theta = 90^o$), then $W = 0$. If **F** has a component opposite to the displacement ($\theta > 90^o$), then the W is negative. Friction always does negative work on a moving object since the frictional force is opposite to the displacement.

More generally for a 3D motion with the displacement $\Delta \boldsymbol{r}$

$$W = \vec{F} \cdot \Delta \vec{r}$$

Example

A 20-kg mass is pulled along a level surface a distance of 4 m by a 150-N force directed 30^o above the horizontal direction. The frictional force acting on the object is 50 N. Find the work done by each individual force acting on the mass and find the total work.



Applied force: $W_F = F\,\Delta x\cos\theta = (150\ N)(4\ m)\cos(30^o)$ = 519.6 J

Normal force: $W_n = n\,\Delta x\cos(90^o)$ = 0

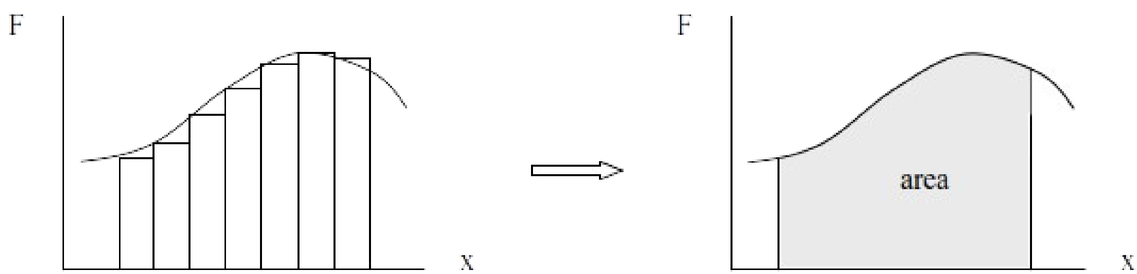
Gravity: $W_g = mg\,\Delta x\cos(90^o)$ = 0

Friction: $W_f = f_k\,\Delta x\cos(180^o) = -f_k\,\Delta x = -(50\ N)(4\ m)$ = -200 J

Total (or net) work: $W = W_F + W_n + W_g + W_n = 519.6\ J - 200\ J$ = 319.6 J

For a force $F(x)$ which varies with position, the net work can be approximated by dividing the displacement into small steps during Δx_i which the force F_i is nearly constant

$$W = \sum_i F_i \Delta x_i$$



In the limit of infinitesimally small steps the graphical interpretation of the work is given by the *area* under the force versus displacement curve. For some simple cases the area can be easily calculate, while in the most general situations calculus is necessary.

Work - Kinetic Energy Theorem

Consider an object of mass m moving a constant velocity v_i , then a constant force F is applied from x_i to x_f such the object accelerates to its final velocity v_f at x_f . Let's calculate the work done by the force as function of the initial and final velocities.

$$W = F\,\Delta x = m\,a\,\Delta x = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

where

$$v_f^2 = v_i^2 + 2\,a\,\Delta x \quad \rightarrow \quad a\,\Delta x = \frac{1}{2}\left(v_f^2 - v_i^2\right)$$

has been used since the acceleration is constant.

Kinetic energy

The term ½ mv^2 is refereed and the *Kinetic Energy* (*KE*)

$$KE = \frac{1}{2}mv^2$$

The kinetic energy is the mechanical energy of an object of mass *m* moving at speed *v*. It is a form of mechanical energy since is the ability to do work because of the state of motion of the object. For example think of a mass m_1 moving at speed *v* colliding against a second mass m_2. As the collision happens a force acts on m_2 over some distance, and so m_1 has the ability to do work, hence *KE*.

Note that the *KE* is not an intrinsic propriety of the object since it depends on the relative motion with the observer. If an observer moves with equal velocity then $v = 0$ and the *KE* of the object with respect to this observer is zero.

Given the definition of *KE* we have the Work – Kinetic Energy Theorem

$$W = \Delta KE$$

Where ΔKE is the change between the final and initial values of the *KE*.

It can be shown that the theorem holds also in the case of a non constant force. From a mathematical prospective the theorem provides a power tool for the calculation of the work done by force. For many physical situations it simpler to measure (or calculate) the initial and final velocities than figuring out the exact expression of the force.

Example

In the previews example, assume that the block starts with an initial speed of 5 m/s. What is its speed after it has moved 4 m?

$$\tfrac{1}{2} mv_f^2 = W_{net} + \tfrac{1}{2} mv_i^2 = 319.6 \text{ J} + \tfrac{1}{2} (20 \text{ kg})(5 \text{ m/s})^2 = 319.6 \text{ J} + 250 \text{ J} = 569.6 \text{ J}$$

$$\tfrac{1}{2} (20 \text{ kg})v_f^2 = 569.6 \text{ J}, \quad v_f = 7.5 \text{ m/s}$$

Potential Energy

The potential energy (*PE*) of a mechanical system is related to the position. That is, it is the ability of a object to do work because of its state of rest at some position.

Gravitational potential energy

An object under that influence of a gravitational force will have potential energy relative to some fixed position. If an object falls a distance $\Delta y = h$ then the work done by the gravitational force F_G is

$$W = \vec{F}_G \cdot \Delta \vec{y} = F_G \cdot \Delta y \cdot \cos 0 = mgh$$

The term cos 0 = 1 is explicitly written to indicate that F_G has same direction of the vertical displacement Δy. The change in potential energy is defined as the negative of the work done by the force. The *PE* is often indicated also as *U*.

$$\Delta U = -W$$

In the case of free falling $\Delta U = -mgh$ which indicates that the potential energy decreases. If the object is moved upward by an external force its potential energy increases and ΔU is positive.

The potential energy is always defined relative to some fixed position. One can set $U_i = 0$ corresponding to the object being on the ground, and the potential energy U_f at height *h* from the ground is given by

$$U_G = mgh$$

Example

A rope pulls a 0.5 kg object up over a distance of 3m. What is the change in its potential energy?

By setting *U_i* = 0

$$\Delta U = U_f - U_i = mgh = (0.5\,kg)(9.89\,m/s^2)(3.0\,m) = 14.7\,J$$

Conservation of Mechanical Energy

The total mechanical energy E_M of a system is the sum of the kinetic and the potential.

$$E_M = KE + PE$$

Consider an object in free the fall: as the object accelerates toward the ground its *PE* decreases while its *KE* increases as it. Using the work-kinetic energy theorem

$$\Delta PE = -W = -\Delta KE$$

or

$$\Delta PE + \Delta KE = \Delta E_M = 0$$

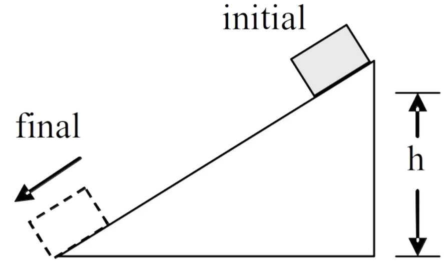
or

$$E_M(initial) = E_M(final)$$

which means that the total mechanic energy does not change. This is the principle of *conservation of energy*.
This principle holds as long as the mechanical system in *closed*. A system is closed if no energy may be transferred into or out of the system, such that its total mechanical energy stays constant.
If friction is present the mechanical system is not closed since some of the mechanical energy is converted into a different form of energy called *Heat* which we will study later in thermodynamics.

Example

A block slides from rest down a friction-less inclined plane of height *h*. What is its speed at the base of the incline?



$E_{initial} = E_{final}$

$K_i + U_i = K_f + U_f$

$\frac{1}{2} mv_i^2 + mgy_i = \frac{1}{2} mv_f^2 + mgy_f$

Choose the base of the incline as $y = 0$. (The choice doesn't really matter as long as we are consistent.) Then $y_i = h$ and $y_f = 0$ and

$0 + mgh = \frac{1}{2} mv_f^2$ and so $v_f = \sqrt{2gh}$

This speed is the same as would be obtained if the mass were dropped straight down from a height *h*. All that matters is the change in height. (Of course the time to slide down would be longer than the time to fall.)

Example

A projectile is fired into the air with speed v_0. What is its speed when it reaches the height *h*? Using conservation of energy,

$$E_i = E_f$$

$$½ m v_f^2 + m g y_f = ½ m v_i^2 + m g y_i$$

Canceling out *m*, we can rewrite this as

$$v^2 = v_0^2 - 2g(y_f - y_i) = v_0^2 - 2gh$$

or, $$v = \sqrt{v_0^2 - 2gh}$$

This is the same result we would have gotten using the projectile equations that were developed using Newton's Laws.

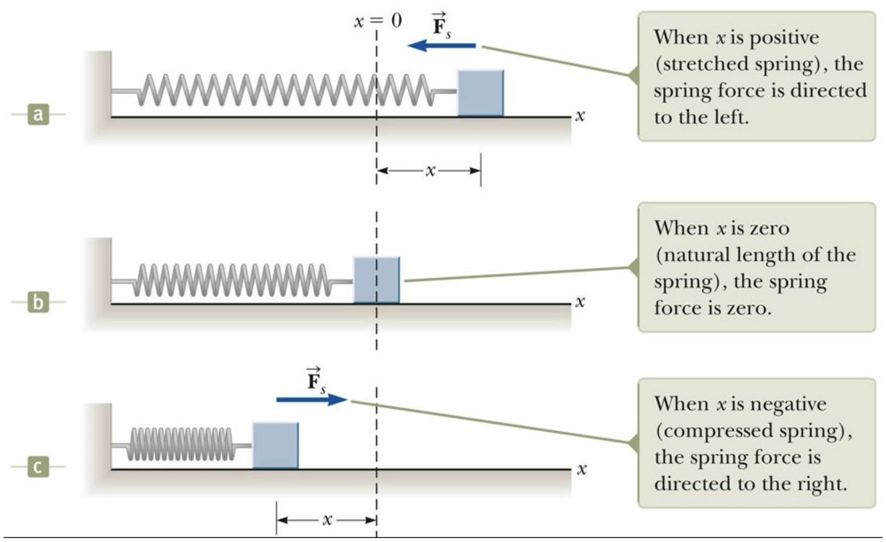
$$v_x = v_{x0} \ , \ v_y^2 = v_{y0}^2 - 2g\,\Delta y$$

$$v_x^2 + v_y^2 = v_{x0}^2 + v_{y0}^2 - 2g\,\Delta y$$

or, $$v^2 = v_0^2 - 2g\,\Delta y$$

Potential Energy Stored in a Spring

Force due to a spring



F_S is the restoring force of the spring, x is the stretch (or compression) of the spring from its equilibrium position, and k is the force constant of the spring (a measure of the spring stiffness). The minus sign signifies that the restoring force is opposite to the displacement.

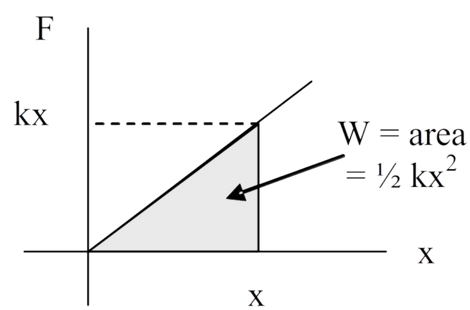
Many springs obey Hooke's law,

$$\vec{F}_s = -k\,\vec{x}$$

Potential energy of a spring

The spring force is a conservative force and a potential energy can be associated with the stretch or compression of the spring.

The graph to the right shows the force required to stretch a spring (opposite to the restoring force) as a function of the stretch. The potential energy stored in the stretched spring is the work required to stretch it. For a constant force $W = F\,x$. If the force is not constant, then we use the average force. For the spring,



$F_{ave} = ½\,(F_i + F_f) = ½\,(k(0) + kx) = ½\,kx$ and so $W = F_{ave}\,x = ½\,kx^2$

Note: Using the average force is the same as taking the work to be the area under the F versus x curve. Thus

$$U_S(x) = \frac{1}{2}kx^2$$

Example

A spring gun has a force constant k = 200 N/m. It is used to fire a 10-g projectile horizontally. If the spring is compressed 7 cm in the cocked position, what is the speed of the projectile when it leaves the barrel of the gun?

We solve this using conservation of energy. We don't need to consider gravitational potential energy since the height of the projectile is the same in the cocked position and when it exits the barrel.

$$E_f = E_i$$

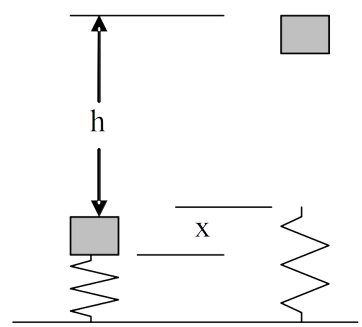
$$\tfrac{1}{2} mv_f^2 + \tfrac{1}{2} kx_f^2 = \tfrac{1}{2} mv_i^2 + \tfrac{1}{2} kx_i^2$$

$$\tfrac{1}{2} mv^2 + \tfrac{1}{2} k(0)^2 = \tfrac{1}{2} m(0)^2 + \tfrac{1}{2} kx^2$$

$$v = x\sqrt{\frac{k}{m}} = 0.07\ m\sqrt{\frac{200\ N/m}{0.01\ kg}} = 9.9\ m/s$$

Example

This same spring gun is fired straight up into the air. How high does it go?



$$E_f = E_i$$

$$\tfrac{1}{2} mv_f^2 + mgy_f + \tfrac{1}{2} kx_f^2 = \tfrac{1}{2} mv_i^2 + mgy_i + \tfrac{1}{2} kx_i^2$$

$$0 + mgh + 0 = 0 + 0 + \tfrac{1}{2} kx^2$$

$$h = \frac{\frac{1}{2}kx^2}{mg} = \frac{\frac{1}{2}(200)(0.07)^2}{(0.01)(9.8)} = 5.0\ m$$

Conservative and Non-conservative Forces

The possibility to define a potential energy depends on the nature of the force. For the gravitational force and the restoring force of a spring is possible to define a potential energy, while for the tension or friction is not. The potential energy can only be associated with a force if that force is *conservative*.

The term conservative refers to the possibility to 'conserve' in the sense of 'preserve or maintain' the mechanical energy of the system.

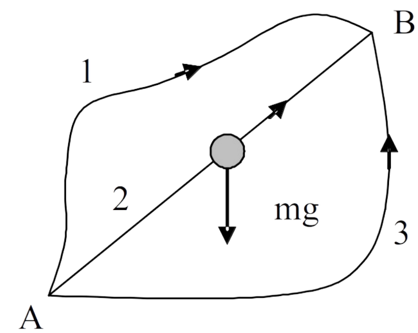
Example

Conservative force: work is done by your hand to lift up an object from your desk to some height *h*. The object gains potential energy. You open your hand and the object falls back on the desk. Its *KE* increases and is recovered as it falls.

Non-conservative force: work if done by your hand to move an object on your desk from one place to another horizontally. There is friction. You open your hand and the object does not return to the initial location. The object did not gain any *PE* from your hand and therefore it does not start to move backward since it has not *PE*.

What happen to the work done by the force applied by your hand?
In the gravitational case it becomes mechanical energy (*PE* and then *KE*)
In the friction case it transforms into heat.

A conservative force is one for which the work done by the force on an object as it goes from point *A* to point *B* doesn't depend on the path taken. The work done by gravity, for example, doesn't depend on path since any horizontal motion is perpendicular to the weight, and doesn't contribute to the work.
In the illustration to the right, the work done by gravity as the object moves from *A* to *B* is the same for paths 1, 2 and 3.



If the object goes in a closed path (from *A* to *B* and back to *A*) then the net work done by the gravitational force is zero.

A non-conservative force is one for which the work depends on the path. In the case of friction if you slide a block on a surface along a closed path the work is not zero.

Non-conservative Forces and Energy

If non-conservative forces are present the mechanical energy is not conserved and we have

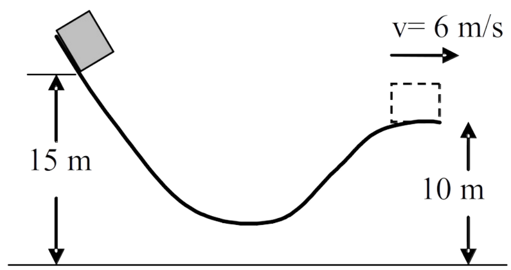
$$\Delta E_M = W_{NC}$$

Where W_{NC} is the net work done by the non-conservative forces. W_{NC} can be positive or negative depending on the nature of the force. For example in the case of friction W_{NC} is negative (frictional forces always act against the direction of motion) and reduces the total mechanical energy.

The expression above is the more general form of conservation of mechanical energy. If conservative forces are present, they are included on the LHS of the equation via their potential energies.

Example

A box of mass 2kg slides down a track as shown in the figure to the right. Its initial velocity at height 15 m is zero, while at height 10 m is 6 m/s. Find the work done by the frictional force.



$W_f = (KE_f + PE_f) - (KE_i + PE_i)$

$= (½\, mv_f^2 + mgy_f) - (½\, mv_i^2 + mgy_i)$

$= [½\, (2)(6)^2 + (2)(9.8)(10)] – [½\, (2)(0)^2 + (2)(9.8)(15)] = 36 + 196 – 294 = - 62\ J$

The work done by friction is negative as expected.

An example of positive W_{NC} is given by the force provided by the thrust of a jet engine. As the jets takes off this W_{NC} increases the total mechanical energy. Assume no air resistance, when the engine is off the jet obeys the laws of a projectile motion, i.e. as it falls down and it accelerates, its *PE* transforms into *KE* and the mechanical energy is conserved. When the engine is on, the jet acquires extra *KE* from the trust force: its final mechanical energy is greater than the initial.

If there are no changes in the potential energy $\Delta PE = 0$ the general form of conservation of mechanical energy $\Delta E_M = W_{NC}$ reduces to

$$\Delta KE = W_{friction}$$

usually referred as the *work-kinetic energy theorem with friction*. For example an object which is moving horizontally then it comes to a stop because of friction.

Power

Mechanical power is the rate at which work is done. If the force applied over the distance is constant then

$$P = \frac{W}{\Delta t} = \vec{F}\cdot\frac{\Delta \vec{x}}{\Delta t} = \vec{F}\cdot\vec{v}$$ Units are J/s = watt (W)

Example

What is the average power required to lift a 60-kg person a height of 2 m in 5 seconds?

$$P=\frac{W}{\Delta t}=\frac{F\Delta y}{\Delta t}=\frac{mg\,\Delta y}{\Delta t}=\frac{(60)(9.8)(2)}{5}=235.2\ W$$

Example

A block of mass m = 280 kg is pulled by a truck driving over a surface with coefficient of kinetic friction μ_k = 0.12 . The block moves at constant velocity v = 10.8 m/s. Find the power necessary for the truck to pull.

Since the block (and the truck) moves at constant speed the net force acting on the block is zero which implies that kinetic friction equals the tension on the rope

$$T = \mu_k mg$$

thus,

$$P = T\cdot v = (0.12)(280)(9.81)\cdot 10.8 = 3560 \quad W$$

Example

The power company bills its customers based on the number of *kilowatts* of energy used. Suppose the energy cost is 10 cents per kwh. How much would it cost to keep a 75 W bulb lamp on for one month?

Δt = (30 days)(24 hr/day) = 720 hr. P = 0.075 kW
E = $P\Delta T$ = (0.075 kW)(720 hr) = 54 kwh

Cost = ($0.10/kwh)(54 kwh) = $5.40