Universal Gravitation

The gravitational interaction is the most fundamental all of interactions since it effects all particles and forms of energy. In Newton’s theory of universal gravitation, gravity is described as an attractive force. The history of the theory of gravity matches with the history of modern science; physics was born as a theory of gravity. Nowadays instead gravity is understood not as a force, but it originates within the geometric proprieties of space and time (see Einstein’s theory of General Relativity).

Galileo

Galileo Galilei discovered experimentally the equivalence principle, i.e. all objects, regardless of their masses accelerates toward the ground at the same rate of 9.81 m/s². Year circa 1580.

Kepler’s Laws of Planetary Motion

Johannes Kepler formulated his three laws of planetary motion in 1609. Kepler believed in the Copernicus planetary system (the planets rotate around the Sun and not the other way around) and deduced these laws by carefully studying the data on planetary motion obtained by Tycho Brahe.

Kepler’s 1st law:

Planets move in elliptical orbits with the Sun at one of the focus points. An ellipse is an oblong closed curve with two focus points, as shown to the right. The ellipse can be traced out by following the path such that the sum of the distances from the focus points to any point on the curve is constant. That is, \( r_1 + r_2 = \text{constant} \) where \( r_1 \) and \( r_2 \) are the distances of \( m \) to \( f_1 \) and \( f_2 \) respectively. In the case of a planet orbiting the Sun, the Sun is located at one of the focus points.

A circle of radius \( R \) is a special case of an ellipse where the two focal points coincide with the center of the circle \( f_1 = f_2 = \text{center} \) and \( r_1 = r_2 = R \).
**Kepler’s 2nd law:**

A line from the Sun to a planet sweeps out equal areas in equal times. As a planet goes in an elliptical path around the Sun, the the time to move from $A$ to $B$ is the same as the time to move from $C$ to $D$. From $A$ to $B$ the planet moves faster than from $C$ to $D$ so that the area swept out by the line from the planet to the Sun is the same for both time intervals.

\[ \frac{\Delta A_1}{\Delta t} = \frac{\Delta A_2}{\Delta t} \quad \text{such that} \quad A_1 = A_2 \]

or

\[ \frac{\Delta A}{\Delta t} = \text{constant} \]

**Kepler’s 3rd law:**

The square of the orbital period of the planets is proportional to the cube of the average distance from the planet to the Sun.

\[ T^2 = Kr^3 \]

where $K = 2.95 \times 10^{-19} \text{s}^2/\text{km}^3$ is the Kepler constant obtained from the data.
**Newton Theory of Universal Gravitation**

Newton formulated the Law of Universal Gravitation. This law states that two objects of mass $m_1$ and $m_2$ at distance $r$ from each other are attracted by a force given by

\[ F = G \frac{m_1 m_2}{r^2} \]

where $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$ is the Newton’s gravitational constant.

For spherical masses, $r$ is the distance between the the center of the mass of the two objects.

There is a force acting on each object: the two forces have the same magnitude $F_1 = F_2 = F$ and opposite directions.

In vector form, with $m_1$ located on the origin, the force acting on $m_2$ is

\[ \vec{F} = -G \frac{m_1 m_2}{r^2} \hat{r} \]

where the negative sigh indicates that the direction of $\vec{F}$ is opposite to $\hat{r}$, that is: the force points toward $m_1$ and is an attractive force.

Newton’s Law of gravity is called universal since it applies to both the gravitational effects on the Earth and the gravitational effects ‘outside’ the Earth (Kepler Laws).

**Equivalence Principle from Newton Law**

The equivalence principle states that all objects, regardless of their mass, accelerate toward the ground at the same rate $g = 9.81 \text{ m/s}^2$. This result follows from the theory of Universal Gravitation.

An object of mass $m$ is located at height $h$ from the surface of the Earth. The object is attracted toward the ground by a gravitational force of magnitude

\[ F = G \frac{M_E m}{r^2} = G \frac{M_E m}{(R_E+h)^2} = G \frac{M_E m}{R_E^2} , \]

where $R_E$ is the radius of the Earth and we used the approximation $r \approx R_E$ since $R_E >> h$. 
The gravitation force is the weight $W = mg$, it follows

$$g = \frac{GM_E}{R_E^2} = \frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(5.98 \times 10^{24} \text{ kg})}{(6.38 \times 10^6 \text{ m})^2} = 9.81 \text{ m/s}^2.$$ 

The numerical value $g$ does not depend on $m$ (equivalence principle). It depends on the mass and radius of the Earth and how far is the object from the center of the Earth. On the Earth’s surface $r = R_E$ and so $g = 9.81$. Even if it is the same for all the objects, its numerical value decrease at higher altitude. At altitude $h = 6.38 \times 10^6 \text{ m}$ above Earth’s surface, $r = 2R_E$ and $g = 9.8/(2^2) = 2.45 \text{ m/s}^2$. A person’s weight would be $\frac{1}{4}$ compared to the weight on the surface.

For the astronauts on the ISS located at altitude of about 400 km the acceleration due to gravity is $0.9g$. That means their weight in $90\%$ of their weight on Earth. The reason they are ‘weightless’ is because they are in free fall, following the trajectory given by the orbital path.

**Kepler Laws from Newton Law**

- **Kepler 1<sup>st</sup> law**

It can be shown mathematically that an object will orbit a second massive object in an elliptical path unless the energy of the orbiting object is so great that the orbit is not closed.

- **Kepler’s 2<sup>nd</sup> law**

It is a direct consequence of conservation of angular momentum and the fact that the force of attraction is directed alone the line connecting the two bodies.

The area swept out by the vector $\mathbf{r}$ from the Sun to the planet in a time $\Delta t$ is

$$\Delta A = \frac{1}{2} r \Delta r = \frac{1}{2} r v \Delta t$$

where $\Delta r = v \Delta t$ is perpendicular to the vector $\mathbf{r}$ connecting the two bodies and $v$ is the tangential velocity (assumed to be constant in order to simplify the calculations).

Thus, $\frac{\Delta A}{\Delta t} = \frac{1}{2} r v$

The angular momentum of the planet about the Sun has magnitude

$$L = rmv$$
Thus,
\[ \frac{\Delta A}{\Delta t} = \frac{L}{2m} \]

The angular momentum of the planet \( L \) is constant since the Sun does not exert a torque on the planet. This is because the force exerted by the Sun is along the line connecting the Sun and the planet.

- Kepler 3\textsuperscript{rd} law

Assuming circular orbit and constant velocity (to simplify the calculations)

\[
F = ma_c
\]

\[
\frac{GM_s m}{r^2} = m \frac{v^2}{r} = m \left( \frac{2 \pi r}{T} \right)^2 = m \frac{4 \pi^2 r}{T^2}
\]

Solving for \( T \)

\[
T^2 = \left( \frac{4 \pi^2}{GM_s} \right) r^3
\]

which provides the numerical values of the Kepler constant expressed using the mass of the Sun

\[
K = \frac{4 \pi^2}{GM_s}
\]

Note: In the case of an elliptical orbit, Kepler’s 3\textsuperscript{rd} law still applies as above except that the radius of the circle is replaced by the semi-major axis of the ellipse.

\textit{Example}

What would be the period of a satellite in orbit just above the surface of earth? (Of course, such an orbit could not be sustained because of atmospheric resistance.)

\[
T^2 = \left( \frac{4 \pi^2}{GM_E} \right) R_E^3 = \left( \frac{4 \pi^2}{(6.67 \times 10^{-11})(5.98 \times 10^{24})} \right) (6.38 \times 10^6)^3
\]

\[ T = 5,070 \text{ s} = 85 \text{ min} \]

What is the speed of a satellite in a circular orbit? From

\[
F = m \frac{v^2}{r} = \frac{GM_E m}{r^2}
\]
we obtain

\[ v = \sqrt{\frac{GM_E}{r}} \]

For such a low Earth orbit,

\[ v = \sqrt{\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.38 \times 10^6}} = 7,900 \text{ m/s} = 17,700 \text{ mph} \]

The same result can be obtained from

\[ v = \frac{2 \pi R_E}{T} \]

Questions: How does the period of orbit change as the radius increases? How does the satellite speed change as the radius changes? How does the period of the orbit depend on the mass of the satellite?

Newton worked backward: he derived the law of universal gravitation from the Kepler laws. To do so he had to invented calculus: the study of continuous changes, in this case applied to a planet which has a varying velocity and distance as it orbits the Sun. Calculus was also invented independently by German mathematician, philosopher, scientists and diplomat (a lot right?) Leibniz, at the same time of Newton.
Gravitational Potential Energy

From the gravitational force formula, one can obtain the expression for the gravitational potential energy $U$, of two attracting masses. As usual, the change in potential energy is defined as the negative of the work $W$ done by the gravitational force to move an object from $r_i$ to $r_f$

$$\Delta U = U_f - U_i = -W_g = -\frac{Gm_1 m_2}{r_f} + \frac{Gm_1 m_2}{r_i}.$$ 

where the work is calculated using the tool of calculus since the force is not constant. For $r_i = \infty$ the potential energy is set to be zero. Thus, an object moved from infinity to a point $P$ located at $r_f = r$

$$U(r) = -\frac{Gm_1 m_2}{r}$$

The expression above can also be interpreted at the potential energy of a system of two masses located at a distance $r$ from each other.

The figure to the right displays the potential energy for a region $r > R_E$. Inside the Earth radius the potential energy has a different expression which we will not consider in this course.

The expression of the gravitational potential energy $PE = mgh$ can be derived from $U(r)$ by setting $r = R_E + h$ and $h << R_E$.

$$U(R_E + h) = -G m M_E \left( \frac{1}{R_E + h} \right) \approx -G m M_E \left( \frac{1}{R_E} - \frac{h}{R_E^2} \right) = -\frac{G m M_E}{R_E} + mgh$$

from which it follows that the potential difference between any two points at a distance apart of $\Delta y = y_f - y_i = h$ is

$$\Delta U = U(y_2) - U(y_1) = \Delta PE = \left[ -\frac{G m M_E}{R_E} + mgh \right]_2 - \left[ -\frac{G m M_E}{R_E} + mgh \right]_1 = mgh$$

The equation $PE = mgh$ is valid only near the Earth’ surface, while $U(r)$ in universal since is valid at any scale. $U(r)$ is always negative while $mgh$ is always positive.
Since only the change in potential energy matters to physics, both expressions agree on the sign of a change of potential energy. If for example a object is raised in gravitation field, the change of potential energy is positive and in both cases the final value is greater than the initial value.

An object of mass $m$ moving in the gravitational field due to mass $M$ has total mechanical energy

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

**Example**

What is the escape velocity of an object from a planet?

By escape velocity, we mean the minimum speed to launch an object such that it never returns to the surface of the planet. This would require that it go an infinite distance from the planet where it eventually comes to rest. We use conservation of energy.

$$E_i = E_f$$
$$K_i + U_i = K_f + U_f$$
$$\frac{1}{2}mv_{esc}^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{r_f} = 0 - 0 = 0$$

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

For Earth, the escape speed is

$$v_{esc} = \sqrt{\frac{2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{6.38 \times 10^6}} = 1.12 \times 10^4 \text{ m/s} \quad (= 25,000 \text{ mph})$$

Note: the escape velocity is the initial velocity, that means no others force are applied after the object takes off. If an object instead has it own propulsion system, as a rocket, then it can escape at any speed. Conservation of energy holds at any distance distance $r > R_E$. After the object takes off with initial velocity $v = v_{esc}$, its velocity and kinetic energy decrease while its potential energy increases, such the total energy stays zero.
Orbits

If the initial velocity of an object is greater than the escape velocity \( v_{\text{esc}} \) its total energy is positive, if it less than its total energy is negative. In this latter case the object is gravitational bound to the planet.

A satellite is released from the space shuttle at some height \( h \) from the Earth’s surface. The initial velocity has direction parallel to the the Earth’s surface. Depending on the magnitude of the initial velocity \( v_0 \) different paths occur:

- If \( v_0 > v_{\text{esc}} \) hyperbola (blue)
- If \( v_0 = v_{\text{esc}} \) parabola (blue)
- If \( v_0 < v_{\text{esc}} \) ellipse (white)
- If \( v_0 = v_{\text{cir}} \) (\(< v_{\text{esc}}\)) circular (white)
- If \( v_0 < v_{\text{cir}} \) ellipse (red)

Blue paths: unbound orbit
White paths: bound orbit
Red paths: bound orbit

Note: If only two celestial bodies are considered, then it is not possible to obtain the orbital paths of procession, of a spiral or scattering.

Circular orbit

Let’s consider circular orbits for simplicity: the orbital velocity is constant and equal to the tangential velocity, \( v_{\text{cir}} = v_{\text{tan}} \).

A satellite of mass \( m_s \) moves in a circular orbit around the Earth, to find its orbital velocity we use Newton second law

\[
\frac{GM_E m_s}{r^2} = m_s \frac{v_{\text{cir}}^2}{r}
\]

Solving for the velocity

\[
v_{\text{cir}} = \sqrt{\frac{GM_E}{r}}
\]
we notice that:

1 - the orbital velocity does not depend on the mass of the satellite (the equivalence principle).
2 - the orbital velocity is smaller for an object orbiting at greater distance (Kepler law).

Comparing with the escape velocity,

\[ v_{cir} = \frac{1}{\sqrt{2}} v_{esc} \approx 0.07 v_{esc} . \]

**Example**

Determine the orbital velocity and period of the International Space Station (ISS) located at altitude \( h = 400 \) km.

\[ v_{cir} = \sqrt{\frac{G M_E}{R_E + h}} = 7.47 \times 10^3 \text{ m/s} \]

for the period

\[ T = \frac{2 \pi (R_E+h)}{v_{cir}} = 5.35 \times 10^3 \text{ s} \]

**Total energy**

In the case of circular orbits, from Newton second law we have

\[ \frac{G m M}{r^2} = m \frac{v^2}{r} \Rightarrow \frac{G m M}{r} = m v^2 \]

Where \( m \) is the mass of the object orbiting around \( M \).

It follows that the kinetic energy can also be written as \( KE = \frac{1}{2} \frac{G m M}{r} \) and the total energy as

\[ E = KE + U = -\frac{1}{2} \frac{G m M}{r} \]

The total energy is negative in agreement with being a bound orbit.
Example

Find the energy necessary to put the Soyuz (the Russian spacecraft of mass $m_s = 9000$ kg which currently brings the astronauts back and forth from the ISS) in the same orbit of the ISS (altitude $h = 400$ km).

The necessary energy is the difference of the final and initial energy

$$\Delta E = E_f - E_i$$

with

$$E_i = U_i = -\frac{Gm_s M_E}{R_E} = -5.53 \times 10^{11} \text{ J}$$

$$E_f = -\frac{1}{2} \frac{Gm_s M_E}{(R_E + h)} = -2.65 \times 10^{11} \text{ J}$$

Once in orbit the kinetic energy of the Soyuz is

$$KE = \frac{1}{2} \frac{Gm_s M_E}{(R_E + h)} = 2.65 \times 10^{11} \text{ J}$$