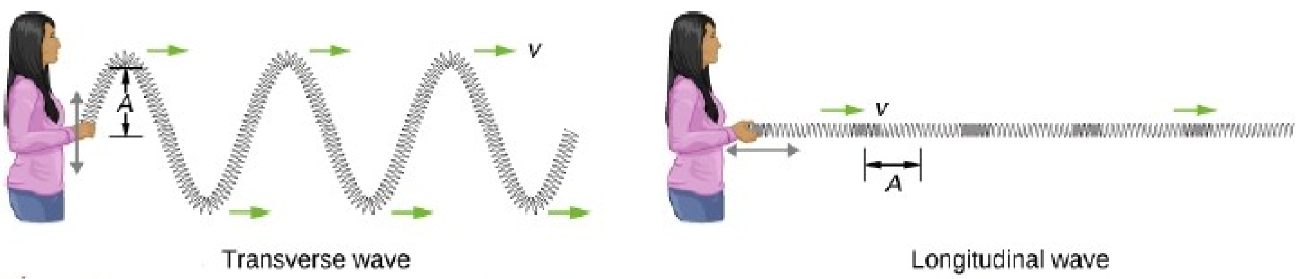
Waves

A wave is energy which propagates in space in the form of a traveling disturbance. If the disturbance travels within a medium the wave is referred as *mechanical wave.* For example sound waves, generated by the vibration of vocal chords, propagate through the air; the ocean waves, generated by the wind hitting the water' surface, propagate through water. Non mechanical waves do not required a medium to propagate through. For example the electromagnetic waves can propagate in empty space (and are generated by charge particle, see PH106).

The waves are classified based how the direction of oscillation which originates the wave, is related to the direction of propagation of the wave.

For example let's consider a long spring and a person moving her hand to generate a wave. The direction of the oscillation (of her hand) is shown by the double gray arrow, the direction of propagation if the wave is the green arrow.



<u>Transverse wave:</u> the oscillation of the hand is perpendicular to the direction of motion.

<u>Longitudinal wave:</u> the oscillation of the hand is along the direction of motion.

Examples of transverse waves are the electromagnetic waves (PH106) and the waves produced in a rope. Sound is a longitudinal wave. There are also waves which are neither traverse or longitudinal but can be describe as a 'combination' of those, as for example the ocean waves.

In this chapter we will focus on transverse waves.

Wave equation

The following wave equation describes a transverse wave oscillating in the *y* direction and traveling in the *x* direction. It is a second order partial differentiate equation

$$\frac{\partial^2 y(x,t)}{\partial^2 t} = v^2 \frac{\partial^2 y(x,t)}{\partial^2 x} \qquad \text{(wave equation)}$$

where *v* is the velocity of the wave. A wave is solution of the wave equation. The most general form of the solution is

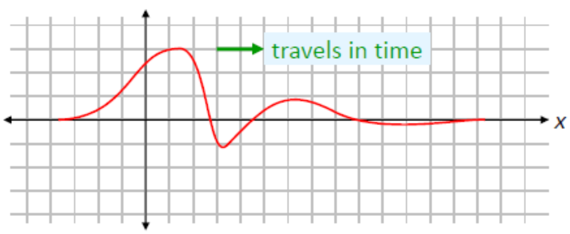
$$y(x,t) = f(x \pm vt) \qquad \text{(wave solution)}$$

That means that any function with argument ($x \pm vt$) represents a wave, with the sign:

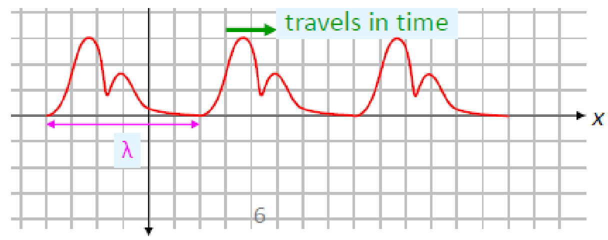
$y(x,t) = f(x - vt)$ if the wave is traveling to the right.

$y(x,t) = f(x + vt)$ if the wave is traveling to the left.

A wave can be a *pulse* or it can be *periodic*



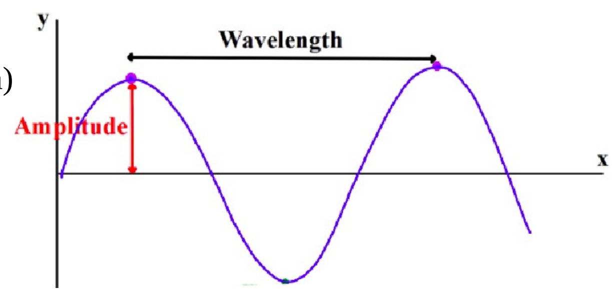
Pulse: wave drops to zero on both sides



Periodic: wave repeats after a period *T*

A *sinusoidal* wave is a periodic wave described by the sine function:

$$y(x,t) = A \sin(kx - \omega t + \varphi)$$



A = amplitude = maximum vertical displacement (m)
$k = 2\pi/\lambda$ = wave number (m^{-1})
λ = wavelength (m)
$\omega = 2\pi f = 2\pi / T$ = angular frequency (rad/s)
f = frequency (Hz)
T = period (sec)
φ = phase angle (depends on choice of $t = 0$)

Note: The sinusoidal wave traveling to the right can also be written as

$$y = A \sin[k(x - \frac{\omega}{k} t) + \varphi] = A \sin[k(x - v t) + \varphi] \,,$$

which has indeed the form $y(x,t) = f(x - vt)$.

The *wave velocity* is the the speed of the crest of the wave moving to the right (or left)

$$v = \frac{\Delta x}{\Delta t} = \frac{\lambda}{T} = \lambda f = \left(\frac{\omega}{2\pi}\right)\left(\frac{2\pi}{k}\right) = \frac{\omega}{k}.$$

The *transverse velocity* refers to the vertical motion *as a function of x and t* (the velocity of the person's hand up and down on the first page). It is obtained by differentiation the transverse displacement.

$$y(x,t) = A\sin(kx - \omega t + \varphi)$$
$$v_y = \frac{dy}{dt} = -\omega\ A\cos(kx - \omega t + \varphi)$$

For a fixed value of *x* the vertical motion $y(t) = A sin(\omega t + \varphi)$ is a simple harmonic motion.

Likewise, the *transverse acceleration* of the wave is given by

$$a_y = \frac{dv_y}{dt} = -\omega^2 A\sin(kx - \omega t + \varphi)$$

Example

A wave is described by the equation $y(x,t) = 0.05\sin(2x - 5\ t\ )$. What are the amplitude, wavelength, period, and wave speed?

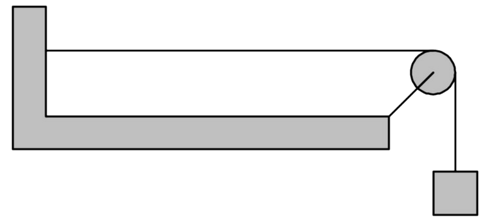
A = 0.05 m
$k = 2$, $\lambda = 2\pi / k = \pi$ = 3.14 m
$\omega = 5$, $T = 2\pi / \omega = 0.4\ \pi$ = 1.26 s
$v = \lambda / T$ = 3.14 m/1.26 s = 2.5 m/s

What is the transverse velocity at x = 1 m and t = 2 s?

$$v_y = -\omega\ A\cos(kx - \omega\ t + \phi) = -(5)(0.05)\cos(2(1) - 5(2))$$
$$= -0.25\cos(-8\,rad) = 0.036\ m/s$$

Wave in a string

A string clamped at one is held in tension by a mass *M*. If a wave travels within the string its velocity is given by



$$v = \sqrt{\frac{T}{\mu}}$$

where *T* is the tension in the rope and μ is the linear mass density of the rope (kg/m).

Example:

A string clamped at one is held in tension by a 1.5-kg weight attached to the other, as shown to the right. The mass of the string is 50 g and the length is 2 m. What is the speed of a wave on this string?

$$F = Mg = (1.5\,kg)(9.8\,m/s^2) = 14.7\ N$$
$$\mu = m/L = 0.05\,kg/2\,m = 0.025\ kg/m$$
$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{14.7\,N}{0.025\,kg/m}} = 24.2\ m/s$$

Power transmitted by a wave

A traveling wave carries energy. If the wave is on a stretched rope, then any piece of the rope moves up and down like a mass on the end of a spring. It has kinetic energy of motion and elastic energy associated with its vertical displacement.
The energy of a piece of rope with infinitesimal *dm* is the sum of its kinetic energy and potential energy.

$$dE = dKE + dPE = \frac{1}{2} v_y^2\, dm + \frac{1}{2}\omega^2 y^2\, dm$$

where we have used the potential energy of a spring with spring constant $k = \omega^2 m$

Since for a simple harmonic motion the potential energy is zero when the kinetic energy is a maximum (and vice versa), the energy is

$$dE = \frac{1}{2} v_{y\,MAX}^2\ dm = \frac{1}{2}(\omega^2 A^2)\mu\, dx$$

where μ is the linear mass density of the rope and *A* is the amplitude of the wave.

The power transmitted is

$$P = \frac{dE}{dt} = \frac{1}{2}\mu\ \omega^2 A^2 \frac{dx}{dt}$$

The wave speed is $v = dx/dt$. Thus, the power transmitted (watts) is

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

Example

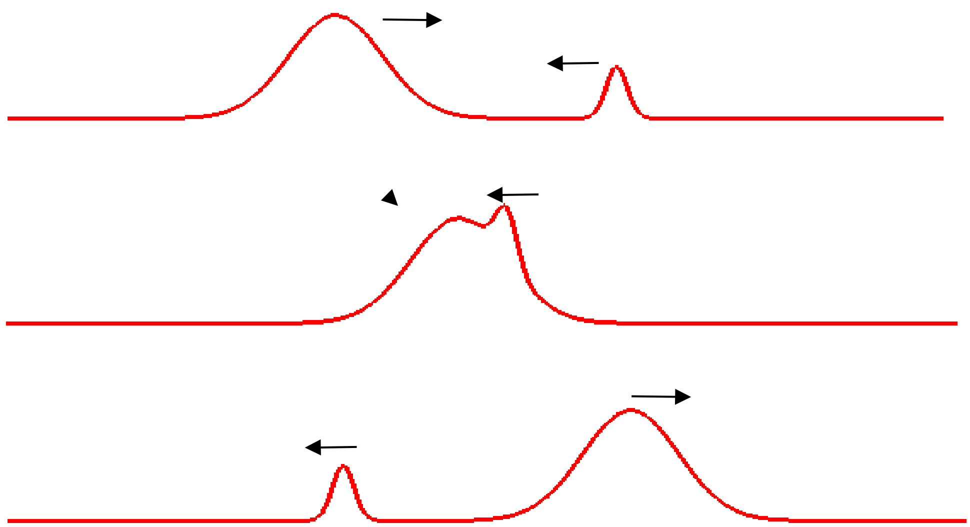
A long rope has a linear density of 0.1 kg/m and subjected to a tension of 40 N. One end of the rope is moved up and down at a rate of 5 Hz with an amplitude of 0.1 m. How much power is transmitted down the rope?

$$v=\sqrt{\frac{40\,N}{0.1\,kg/m}}=20\ m/s$$

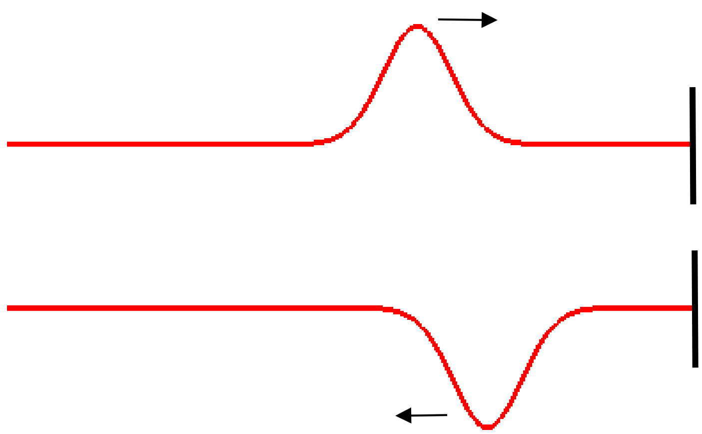
$$P=\frac{1}{2}\mu\ v\ \omega^2 A=\frac{1}{2}(0.1\,kg/m)(20\,m/s)(2\pi(5\,Hz))^2(0.1\,m)^2=9.9\ W$$

Interference and Reflection of Waves

Waves whose amplitudes are not too great obey the principle of superposition. This means that when two waves meet, the resultant wave is obtained by adding the two waves point by point. Waves do not collide and bounce off each other as is the case for particles. They travel through each other and emerge as though they did not interact.

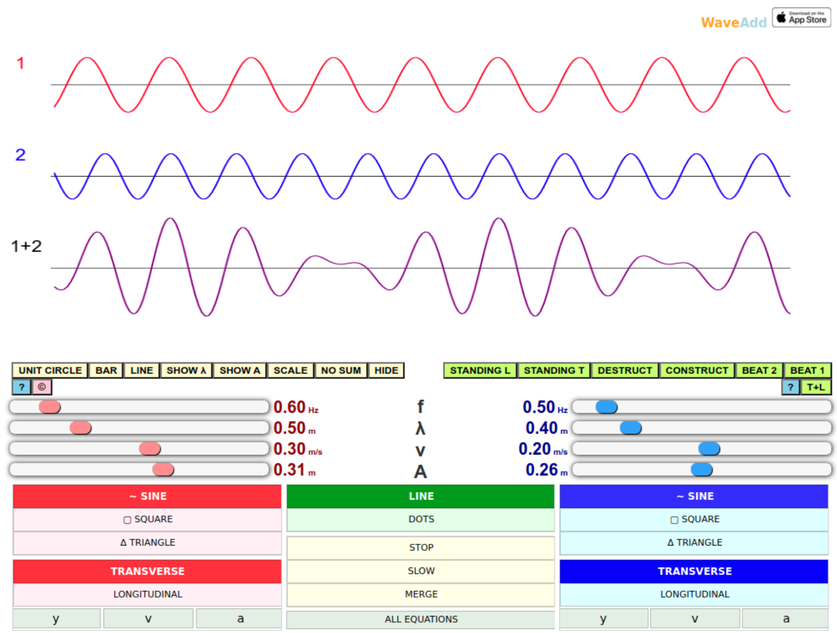


Waves can reflect, however, from the end of a rope. If the end of the rope is tightly secured, then the wave will invert when reflected.



If the end of the rope is untied and loose, then the wave will reflect back without inversion. If two ropes with different mass densities are tied together, then a wave will be partly reflected from the point where they are joined together.

Wave interference physics simulation [link](#)

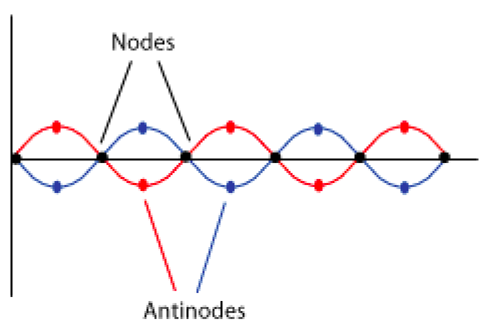


Standing waves on a string

Waves can interfere either constructively or destructively, depending on their relative phase when they combine. If waves are generated on a string that is clamped at both ends, then the waves will be reflected at the clamped ends. The resulting wave will consist of a sum of waves going left and right.

The blue wave travels to the left, it hits the wall and bounces back into the red wave traveling to the right. If the wavelength conditions is satisfied, the wave interference with itself (the blue with the red) and form a *standing wave*.

The locations where the two waves interfere constructively are the *antinodes* and where they interfere destructively are the *nodes.*



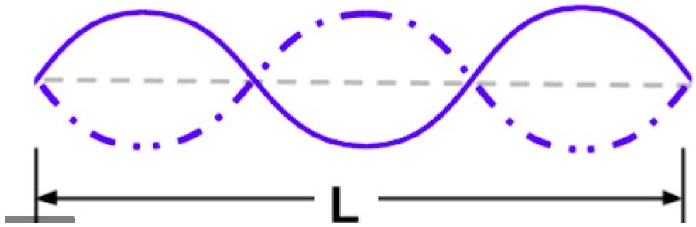
The wavelength condition is

$$\lambda_n = \frac{2L}{n} \qquad n = 1,2,3\ldots$$

where L is the fixed distance between the ends and n is the number of loops.

In the figure to the right
the number of loop is $n = 3$
and the wavelength is

$$\lambda_3 = \frac{2}{3}L$$



Allowed standing wave frequencies:

$$\lambda_n = \frac{2L}{n}$$
$$f = \frac{v}{\lambda} = \frac{v}{2L/n} = n\frac{v}{2L}$$
$$v = \sqrt{\frac{F}{\mu}}.$$

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}},\ n = 1,\ 2,\ 3,\ \ldots$$
$$n = 1:\ f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}} \quad 1st\ harmonic\ (or\ fundamental)$$
$$n = 2:\ f_2 = 2f_1 \quad 2nd\ harmonic\ (or\ 1st\ overtone)$$
$$n = 3:\ f_3 = 3f_1 \quad 3rd\ harmonic\ (or\ 2nd\ overtone)$$
$$etc.$$

Example

A string 0.8 m long that is clamped at both ends has a tension of 100 N and a linear mass density of 0.02 kg/m. What are the three lowest allowed standing wave frequencies?

$$f_n = \frac{n}{2L}\sqrt{\frac{F}{\mu}} = \frac{n}{2(0.8)}\sqrt{\frac{100}{0.02}} = n\ 44.2\ Hz$$
$$f_1 = 44.2\ Hz,\ f_2 = 2f_1 = 88.4\ Hz,\ f_3 = 3f_1 = 132.6\ Hz$$