AC Circuits

This chapter deals with circuits consisting of combinations of resistors, capacitors, and inductors in which the currents and voltages vary sinusoidally with time. The notation is to use lowercase *v* and *i* to indicate voltage and current.

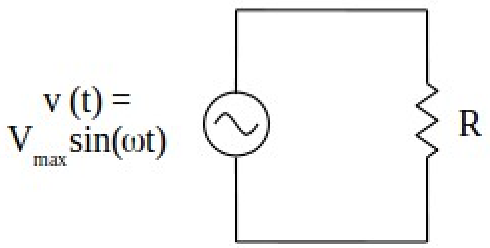
Assume the voltage source is given by

$$v(t) = V_{max}\sin(\omega t),$$

where V_{max} is the peak voltage, $\omega = 2\pi f$ is the angular frequency of oscillation (rad/s), and f is the frequency of oscillation (Hz).

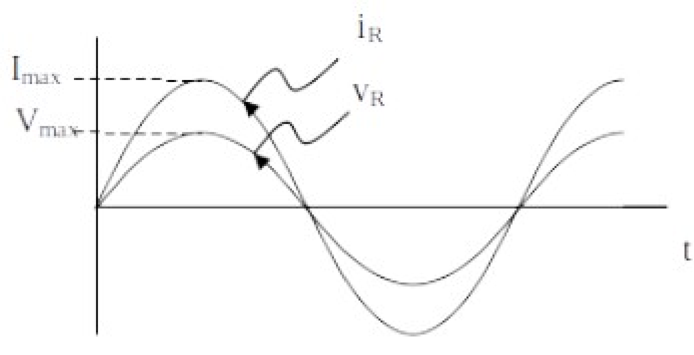
Resistor circuit

If the AC voltage source is connected across a resistor, then the current also varies sinusoidally and is in phase with the voltage.



$$i = \frac{v}{R} = I_{max}\sin(\omega t).$$

$$I_{max} = \frac{V_{max}}{R}$$



The power dissipated in the resistor is given by

$$P = i^2 R$$

Since *i* varies sinusoidally with time, then the power also varies with time. The average power dissipated is

$$P_{avg} = \langle i^2 \rangle_{avg} R$$

The average of $\sin^2(\omega t)$ is ½. So,

$$P_{avg} = \frac{1}{2} I_{\max}^2 R$$

We can write the average power as

$$P_{avg} = I_{rms}^2 R \ ,$$

where

$$I_{rms} = \frac{I_{\max}}{\sqrt{2}} = 0.707 \ I_{\max} \ .$$

I_{rms} is called the 'root-mean-square' current. It is obtained by first squaring the current, then finding the average (mean) of the square, and then taking the square root of this average. It is the effective heating value (or DC value) of the time-varying current. The rms voltage is defined in a similar way –

$$V_{rms} = \frac{V_{\max}}{\sqrt{2}} = 0.707 \ V_{\max}$$

The rms current and rms voltage are related by

$$V_{rms} = I_{rms} R$$

Example

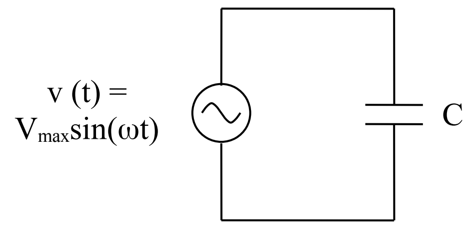
The line voltage in a house is nominally 120 volts rms. What is the maximum (peak) voltage?

$$V_{\max} = \sqrt{2} \ V_{\text{rms}} = \sqrt{2} (120 V) = 170 \ V$$

The *peak-to-peak* voltage is the difference between the maximum positive and maximum negative voltages and is 340 V.

Capacitor circuit

If a sinusoidal voltage source is connected across a capacitor, then the charge on the capacitor and the current to the capacitor also vary sinusoidally with time.



Applying Kirchhoff rule at the one loop

$$v(t) - v_c = 0 \;\Rightarrow\; v(t) = q/C$$

where v_c = q/C is the potential across the capacitor. Writing $q = C\,v(t)$, the current is

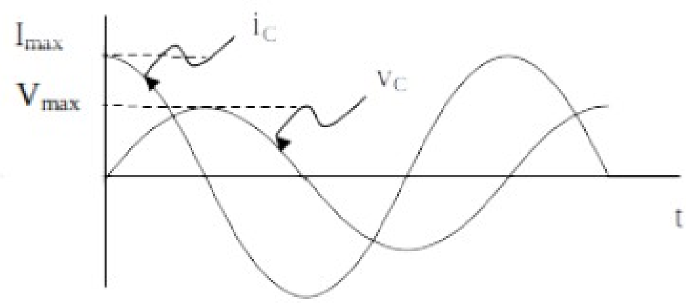
$$i = \frac{dq}{dt} = C\frac{dv(t)}{dt} = CV_{max}\frac{d\sin(\omega\ t)}{dt} = \omega\ CV_{max}\cos(\omega\ t)$$

or

$$i = I_{\max}\sin(\omega\ t + \pi/2)\ ,$$

where

$$I_{\max} = \omega\ CV_{max}\ .$$



For the capacitor, we can write the peak current-voltage relationship as

$$V_{\max} = I_{\max} X_C\ ,$$

where

$$X_C = \frac{1}{\omega\ C} = \frac{1}{2\pi\ f\ C}$$

X_C is called the *capacitive reactance* and has units of ohms. It is the quantity that limits the current to the capacitor, similar to resistance. However, unlike resistance, power cannot be dissipated in a capacitor. This is because the current and voltage are 90^o out of phase. This is analogous to pushing on a moving object. No work will be done by the force if it is applied perpendicular to the displacement.

Note that the capacitive reactance decreases with increasing frequency and capacitance.

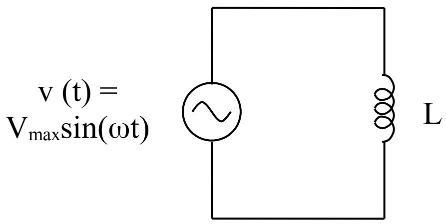
Example

A 0.02 μF capacitor is connected to a 50-V peak AC voltage source which oscillates at 10 kHz. What is the peak current to the capacitor?

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi (1x10^4 Hz)(0.02x10^{-6} F)} = 796\ \Omega$$

$$I_{max} = \frac{V_{max}}{X_C} = \frac{50\ V}{796\ \Omega} = 0.063\ A$$

Inductor circuit



Applying Kirchhoff rule at the one loop

$$v(t) + v_L = 0 \Rightarrow v(t) = L\frac{di}{dt}$$

where v_L = - *L di/dt* is the potential across the inductor. The current is given by

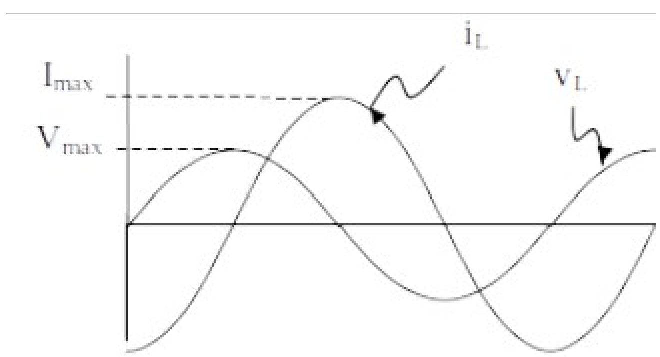
$$i = \frac{1}{L}\int v dt = \frac{V_{max}}{L}\int \sin\omega\ t dt = -\frac{V_{max}}{\omega\ L}\cos\omega\ t$$

Or

$$i = I_{max}\sin(\omega\ t - \pi/2)$$

where

$$I_{max} = \frac{V_{max}}{\omega\ L}$$



This means that in an AC circuit, the voltage is sinusoidal and *leads* the current by 90°.

The peak current-voltage relationship for an inductor can be written as

$$V_{\max} = I_{\max} X_L \ ,$$

where

$$X_L = 2\pi\ f\ L$$

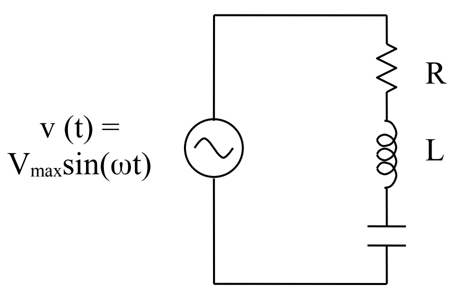
is called the *inductive reactance*.

As with the capacitor, no power can be dissipated in an ideal inductor (one with no resistance).

Note that the inductive reactance increases with increasing *f* and *L*.

Series LCR circuit

Consider an AC circuit containing a resistor, capacitor, and inductor in series. All have the same current (which is true for any series circuit). As previously described, the voltage across the resistor is in phase with the current, the voltage across the capacitor lags the current by 90°, and the voltage across the inductor leads the current by 90°. This means that the voltages across the inductor and capacitor are 180° out of phase. That is, they subtract. The resulting voltage across the inductor and capacitor combination either leads or lags the voltage across the resistor, depending on whether V_L is greater than or less than V_C.



Consequently, the peak voltage across the inductor-capacitor combination is

$$V_{LC,\max} = I_{\max} |X_L - X_C|$$

Since the voltage across the LC combination is 90° out of phase with the voltage across the resistor, then the total peak voltage must be obtained using the Pythagorean theorem –

$$V_{\max} = I_{\max} \sqrt{R^2 + (X_L - X_C)^2}$$

The *impedance* of the circuit is given as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \,,$$

So we can write

$$V_{\max} = I_{\max} Z \;.$$

Z has units of ohms and is a measure of the resistance of the circuit to the flow of current. The reactance is a special case of impedance when the phase difference between current and voltage is either $\pi/2$ or $-\pi/2$.

The total current and the power dissipated in a series RLC circuit depend on the phase shift between the total current and the total voltage. This phase shift depends on the ratio of the out-of-phase to the in-phase voltage. Thus,

$$\tan\varphi = \frac{V_{LC}}{V_R} = \frac{I_{\max}(X_L - X_C)}{I_{\max} R} \;,$$

or,

$$\tan\varphi = \frac{X_L - X_C}{R}$$

If $X_L = X_C$, then the total phase shift is zero and we get maximum current and maximum power dissipated in the resistor. (No power is dissipated in the inductor and the capacitor.)

<u>Resonance in a series LCR circuit</u>

The total current in the LCR circuit is given by

$$I_{\max} = \frac{V_{\max}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

Since X_L and X_C depend on frequency, then I_{rms} depends on frequency.

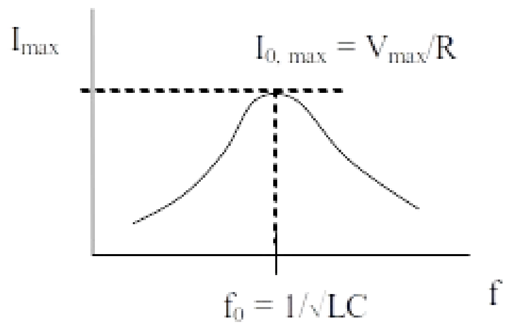
There is a particular frequency for which $X_L = X_C$, at which I_{max} (and I_{rms}) has its maximum value. At this resonance frequency the voltages across the inductor and capacitor exactly cancel, and the entire voltage drop is across the resistor. A plot of the current as a function of frequency would look something like the following.

The resonance frequency, f_0, is given by

$$X_L = X_C$$
$$2\pi f_0 L = \frac{1}{2\pi f_0 C}$$

or,

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$



Example

Consider an RLC circuit for which R = 10 Ω, L = 0.2 mH, C = 5 μF and the applied voltage is V_{rms} = 25 V?

What is the resonance frequency?

$$f_0 = \frac{1}{2\pi\sqrt{(0.2 x 10^{-3} H)(5 x 10^{-6} F)}} = 5,033\ Hz = 5.033\ kHz$$

What would be the current in the circuit if f = 3 kHz?

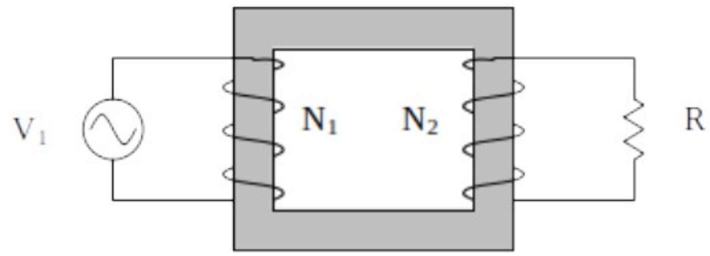
$$X_L = 2\pi fL = 2\pi(3 x 10^3)(0.2 x 10^{-3}) = 3.77\ \Omega$$
$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(3 x 10^3)(5 x 10^{-6})} = 10.6\ \Omega$$
$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(10)^2 + (3.77 - 10.6)^2} = 12.1\ \Omega$$
$$I_{rms} = \frac{V_{rms}}{Z} = \frac{25\ V}{12.1\ \Omega} = \underline{2.06\ A}$$

What is the power dissipation in the circuit?

$$P_{avg} = I_{rms}^2 R = (2.06)^2 (10) = 42.6\ W$$

Transformers

A transformer consists of two coils which are closely coupled so that the flux generated by one coil (the primary) passes mostly through the other coil (the secondary). The flux coupling can be made nearly complete if the coils are wound around an easily magnetizable core such as iron.



According to Faraday's law, the primary voltage is given by

$$V_1 = -N_1 \frac{d\Phi_B}{dt}$$

and the secondary voltage is given by

$$V_2 = -N_2 \frac{d\Phi_B}{dt}$$

Thus, we have

$$V_2 = \frac{N_2}{N_1} V_1$$

The secondary voltage can thus be larger or smaller than the primary voltage, depending on the turns ratio. If we assume that the power delivered into the primary is the same as the power delivered to the load by the secondary,

$$P_1 = P_2$$
$$I_1 V_1 = I_2 V_2$$

then we find that

$$I_2 = \frac{N_1}{N_2} I_1$$

That is, a transformer which steps up the voltage must step down the current, and vice versa.

Example

A transformer has 20 primary turns and 100 secondary turns. If the primary voltage is 12 V, what is the secondary voltage?

$$V_2 = \frac{N_2}{N_1} V_1 = \left(\frac{100}{20} \right) (12\ V) = 60\ V$$

If the load resistor on the secondary is 50 Ω, then

$$I_2 = \frac{V_2}{R} = \frac{60\ V}{50\ \Omega} = 1.2\ A$$

The current in the primary is then

$$I_1 = \frac{N_2}{N_1} I_2 = \left(\frac{100}{20} \right) (1.2\ A) = 6\ A$$