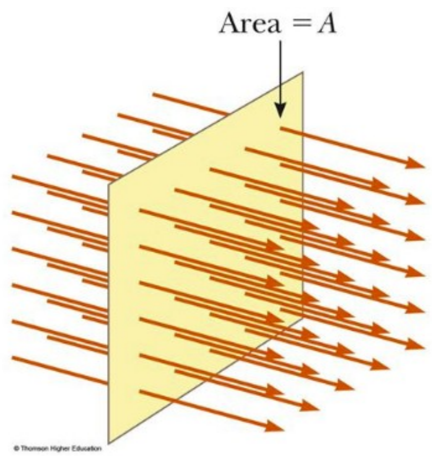
Gauss' Law

Electric Flux

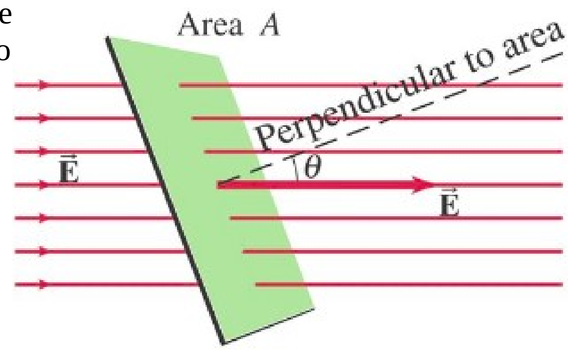
We previously stated that the *electric field strength* is proportional to the density of the electric field lines. That is, the number of lines per unit area perpendicular to the lines. The *electric flux* is defined as the field strength times the perpendicular area and is proportional to the number of lines that go through the area. If the area is perpendicular to **E**, the flux is defined as

$$\Phi_E = EA \quad \text{(units are Nm}^2\text{/C)}$$



If the area is not perpendicular to **E**, then one needs to take the component of **E** that is perpendicular to A. Surfaces do not have directions, what it can be done is to define the direction of a surface by defining the versor **n** which is perpendicular to the surface at given point. Then one obtains the vector quantity **A** = **n**A.

$$\Phi_E = \vec{E}\,\vec{A} = \vec{E}\,\hat{n}\,A = EAcos\theta$$

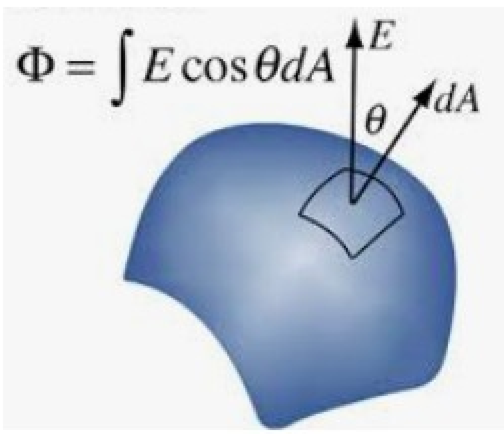


For a continuous surface with a field that varies over the surface, the above would be generalized to

$$\Phi_E = \int \vec{E}\cdot d\vec{A}$$

where

$$\int \vec{E}\cdot d\vec{A} \;=\; \int \vec{E}\,\hat{n}\,dA \;=\; \int E\cos\theta\; dA$$



Gauss' Law

If the surface of the flux is a *closed* surface then we have the Gauss Law. A closed surface is surface which contains a finite volume and has no boundary. No boundary means that you can walk on this surfaces and never find an end. Examples of closed surfaces are cubes and spheres.

For a closed surface, flux can be positive or negative. If the field lines come out of the surface, then the flux is positive. If they go into the surface, then the flux is negative. *Gauss' Law* related the net flux leaving or entering a closed surface to the net amount of charge contained in that surface. Specifically, Gauss's law is

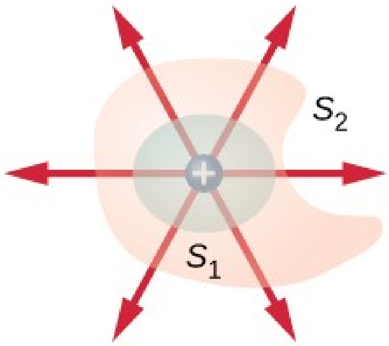
$$\Phi_E = \frac{Q_{enclosed}}{\varepsilon_0}$$

The quantity $Q_{enclosed}$ is the net charge contained inside a Gaussian surface. The constant ε_0 is called the 'permittivity of free space' and is related to the Coulomb constant k_e by

$$\varepsilon_0 = \frac{1}{4\pi\ k_e} = 8.85\,x\,10^{-12}\,C^2/N\cdot m^2$$

The amount of flux depends only on $Q_{enclosed}$ and not the shape or size of the surface. In the figure on the right the two surface S_1 and S_2 enclose the same positive point charge q. The two fluxes are

$$\Phi_1 = \frac{q}{\varepsilon_0} \quad \text{and} \quad \Phi_2 = \frac{q}{\varepsilon_0}$$



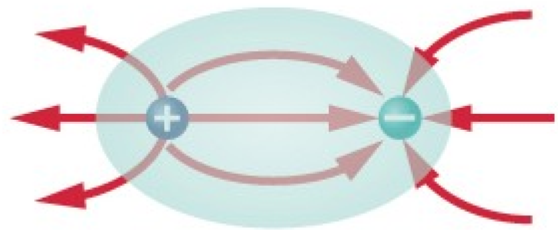
which are the same. The reason the fluxes are equal is because even if S_2 is greater it is also located where the electric filed is weaker (E is a more far away from the source q).

The primary importance of the Gauss's Law is due to the fact that is the first of a set of four equations called the Maxwell's equations.

Those four equations contain altogether the information necessary to derive all the results in electromagnetism and optics. These four equations were discovered by more than one person, but it was J.C. Maxwell to realize the importance of this specific set of four equations, among the many equations in the theory of electromagnetism.

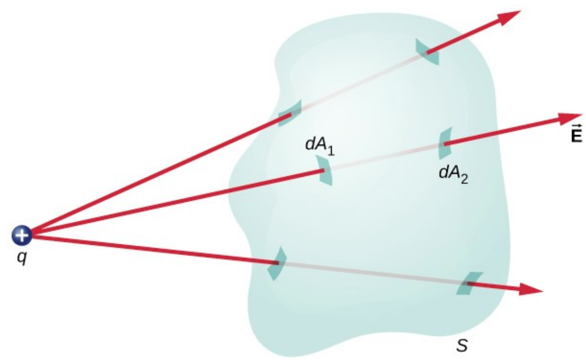
Enclosed Charge (examples)

1 - Given the two point charges of opposite signs and the Gaussian surface in the figure to the right $Q_{enclosed} = q + (-q) = 0$ and so the flux is zero.
If for the same configuration of two charges the Gaussian surface was surrounding only the negative charge the the flux would be

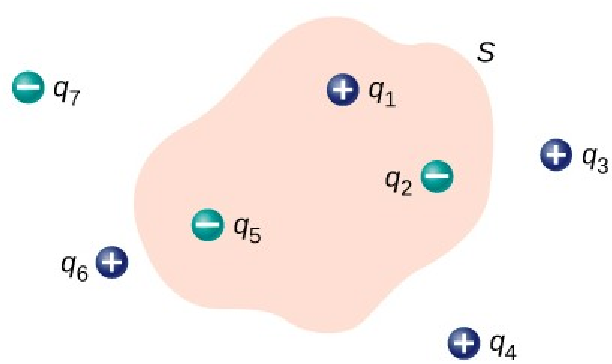


$$\Phi_E = -\frac{q}{\varepsilon_0}$$

2 - Given the one point charge q and the Gaussian surface in the figure to the right $Q_{enclosed} = 0$ since the surface in not enclosing the given charge q. The flux is zero.



3 - The Gaussian surface encloses only the three charges q_1 q_2 and q_5



$$Q_{enclosed} = q_1 + q_2 + q_5$$

so the flux is

$$\Phi_E = \frac{q_1 + q_2 + q_5}{\varepsilon_0}$$

The contribution of the other charges is zero since they are outside the Gaussian surface.

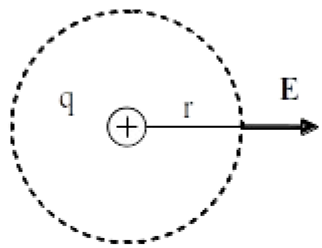
Electric field due to symmetric charge distribution

The Gauss's Law has several applications. It can be use to find the electric filed due to a charge distribution in a much simple way than the integrate the charge $\vec{E} = k_e \int \frac{dq}{r^2}\hat{r}$.
To use the the Gauss's Law the charge distribution requires some degree of symmetry.

<u>Electric Field due to a Point Charge</u>

We can show that Gauss' law applies for a point charge at the center of a spherical surface.

$$Q_{inside} = q = \varepsilon_0 \Phi_E = \varepsilon_0 EA = \varepsilon_0 E 4\pi\, r^2$$



Solving for E,

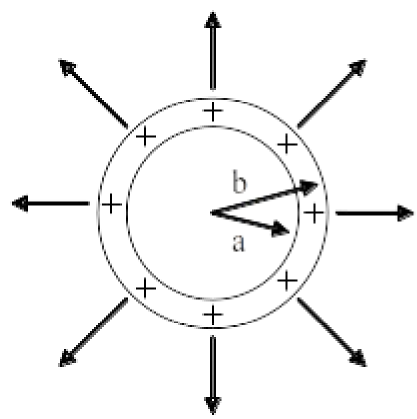
$$E(r) = \frac{1}{4\pi\,\varepsilon_0}\frac{q}{r^2} = k_e \frac{q}{r^2}\ .$$

This is the same as what we previously determined using Coulomb's law.

<u>Electric Field of a Spherical Shell of Charge</u>

An overall charge *q* is present on a spherical shell. Find the electric field.
Outside: consider an imaginary spherical surface outside the shell and concentric with the shell. We can then apply Gauss' law just as we did above for a point charge and conclude that

$$E_{outside} = k_e \frac{q}{r^2}$$

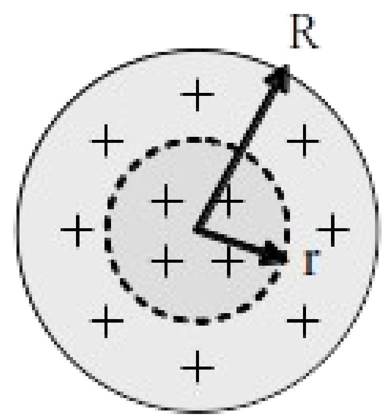


Inside: consider an imaginary concentric spherical surface inside the shell ($r < a$), then we conclude that $E_{inside} = 0$ since there is no charge inside this spherical surface.

If the shell is a metal, then we can further conclude that for $a < r < b$ then $E = 0$ since this must always be the case for a conductor in electrostatic equilibrium.

Electric Field of a Uniform Sphere of Charge

The electric field outside a uniform sphere of charge *Q* and radius *R* is the same as though the charge were a point at the center of the sphere. However, inside the distribution, the electric field depends on position. At a distance $r < R$ from the center, the electric field depends on the charge *Q'* enclosed by a sphere or radius *r*. Since ρ is constant



$$\frac{Q}{V} = \rho = \frac{Q'}{V'} \quad \text{from which}$$

$$Q'(r) = \rho\ V' = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\tfrac{4}{3}\pi r^3\right) = \frac{Q}{R^3}\ r^3$$

Thus, applying Gauss' law we have

$$Q' = \varepsilon_0 \Phi_E = \varepsilon_0 \oint E \cdot dA = \varepsilon_0 E\, 4\pi\ r^2$$

$$E(r) = \frac{1}{4\pi\ \varepsilon_0}\frac{Q'}{r^2} = \frac{1}{4\pi\ \varepsilon_0}\frac{Q\ r^3/R^3}{r^2} = \frac{1}{4\pi\ \varepsilon_0}\frac{Q}{R^3}\, r$$

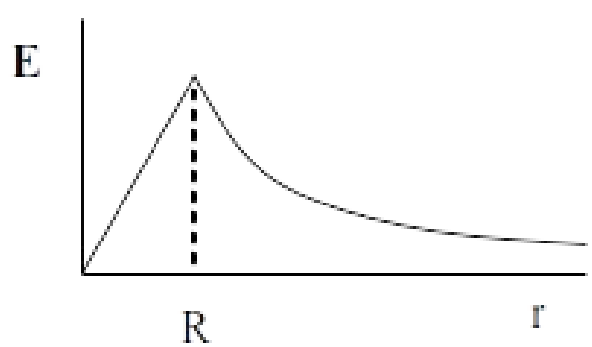
Thus, $E = 0$ at the center, it increases linearly with increasing *r* and reaches a maximum at the surface when $r = R$.

For $r > R$, using the Gauss law it follows:

$$\int E(r)dr \ = \ \frac{Q}{\epsilon_0} \quad \rightarrow \quad E(r)4\pi r^2 \ = \ \frac{Q}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}$$

which is the same electric filed due to a point charge *Q*, as the entire sphere was shrunk into a point.

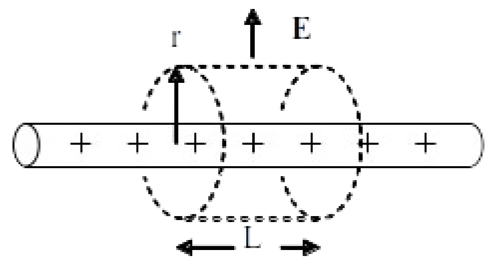
The plot on the right displays The electric field inside the and outside a sphere of radius *R*



Electric Field of a Long Line of Charge

The electric field due to a long line of charge can be determined using Gauss' law by considering an imaginary concentric cylindrical surface containing a portion of the line of constant charge density $\lambda = q/L$.

If the line of charge is very long compared with the length of the 'Gaussian' cylindrical surface, then by symmetry **E** is everywhere perpendicular to the curved part of the surface. There is no flux through the ends of the cylindrical surface since **E** is parallel to the end surfaces. The area of the curved part of the cylinder is $A = 2\pi r L$. Thus,



$$Q_{inside} = q = \varepsilon_0 \Phi_E = \varepsilon_0 EA = \varepsilon_0 E 2\pi r L$$

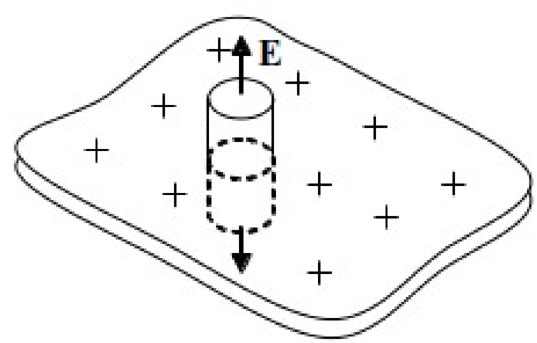
Or,

$$E(r) = \frac{q/L}{2\pi \varepsilon_0 r} = 2k_e \frac{q/L}{r} = 2k_e \frac{\lambda}{r}$$

The field is thus proportional to the charge per unit length (q/L) and inversely proportional to the distance from the center of the line of charge.

Electric Field of a Plane of Charge

The electric field due to a large sheet of uniform charge can be obtained by considering a Gaussian cylindrical surface containing a portion of the charge. If the sheet is large compared with the size of the Gaussian surface, then by symmetry **E** is perpendicular to the surface. The flux coming from the Gaussian surface comes from the top and bottom parts, which each have area A_0. Thus, the total flux is



$$\Phi_E = (\Phi_E^A + \Phi_E^B) = EA_0 + EA_0 = 2EA_0$$

Applying Gauss' law, we have

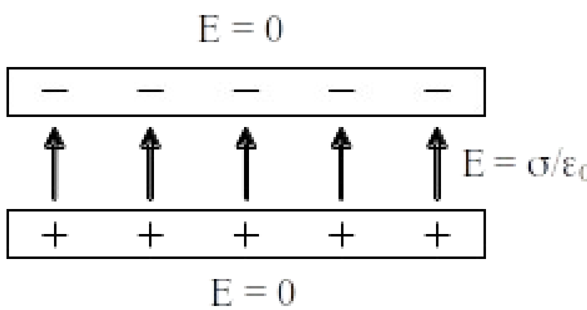
$$Q' = \varepsilon_0 \Phi_E = 2\varepsilon_0 EA_0, \quad or \quad E = \frac{Q'/A_0}{2\varepsilon_0}$$

1. Defining the charge per unit surface as $\sigma = Q/A = Q'/A_0$ we have
$$E = \frac{\sigma}{2\varepsilon_0}$$

Note that E does not depend on the distance from the surface. This is because the surface is assumed to be infinitely large and looks the same regardless of the distance (like saying that the acceleration of gravity is constant as long as the distance from the surface of earth is not too great – in which case the earth looks flat.)

Electric field due to two charged parallel plates

Two parallel metal plates with opposite charges. The electric field between, above, and below the plates is just the superposition of the field due to an infinite plane of positive charge and an infinite plane of negative charge, taking into account the directions of the fields in the different regions.



The fields above and below the plates cancel each others

$$E = 0$$

The fields between the plates add

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

The net electric field between the two plates is uniform (same magnitude and direction).

The given system of the two charged parallel plates is refer as a parallel plate *capacitor*. We will see later in the course that a capacitor is an important element in electrical circuits.