Electric Potential

This chapter covers the electric potential energy and the electric potential difference.

Work done by an electric field

A charge in an electrical field experiences a force given by $\boldsymbol{F} = q\boldsymbol{E}$. If the charge moves, then the force does work. If $\boldsymbol{E}$ is uniform and has a component in the direction of the displacement, then the work done by the force is

$$W = F_x \Delta x = qE_x \Delta x$$

Loosely speaking the work is referred as *the work done by the field.*
If $\boldsymbol{E}$ makes an angle θ with respect to the displacement, then this can be written as

$$W = F\Delta x \cos\theta = qE\,\Delta x \cos\theta$$

More generally, if **E** is not uniform, then

$$W = \int F \cos\,\theta\ dr = q\int E\cos\,\theta\ dr = q\int \vec{E}\cdot d\,\vec{r}$$

The electrical force is conservative, so we can associate a potential energy with this force. If the work done by $\boldsymbol{E}$ is positive, then the potential energy decreases; if it is negative, then the potential energy increases. Think about the analogy with the gravitation force: when an object is in free fall the gravitational force is in the same direction of motion and therefore the work done is positive, the potential energy decreases. This implies that change of potential energy ΔPE equals the negative of the work done.

Specifically if $\boldsymbol{E}$ is constant, then

$$\Delta PE = -W = -qE_x \Delta x$$

More generally

$$\Delta PE = -W = -q\int \vec{E}\cdot d\,\vec{r}$$

According to the work-energy theorem, $W = \Delta KE$. This means that the total energy (kinetic plus potential) is constant.

$$\Delta KE + \Delta PE = 0$$

Or,

$$KE + PE = constant$$

Example

A proton is placed at rest between oppositely charged parallel plates where the electric field strength is E = 500 N/C. What is the speed of the proton after it has moved 1 cm?

$$W = qE\Delta x = (1.6\,x\,10^{-19}C)(500\,N/C)(0.01\,m) = 8\,x\,10^{-19}\,J$$
$$\Delta KE = \frac{1}{2}mv^2 = W$$
$$v = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(8\,x\,10^{-19}J)}{(1.67\,x\,10^{-27}kg)}} = 3.1\,x\,10^4\,m/s$$

Electric Potential Difference Δ*V*

The *electric potential difference ΔV* is the difference in the values of the electric potential between two points *A* and *B*.
Given a charge distribution *ρ* which produces an electric field ***E*** in a region of space, in order to define *ΔV* between two points, consider another charge *q* and the change in the electric potential energy *ΔPE* of the charge as it moves from one point to the other. The electric potential difference is defined as

$$\Delta V = V_B - V_A = \frac{\Delta PE}{q}$$ (unit = J/C = volt = V)

In the case of uniform *E* since $\Delta PE = -qE_x\Delta x$, then

$$\Delta V = -E_x\Delta x$$

Or, in the general case

$$\Delta V = -\int \vec{E}(\vec{r})\cdot d\vec{r}$$

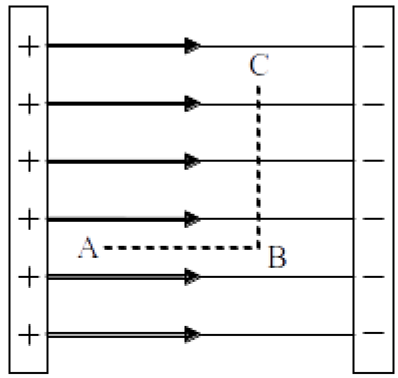
The electric potential difference and the electric potential energy are two distinct concepts. Usually they are referred as the *potential* and the *potential energy* dropping the terms *electric* and *difference.*
The potential is a way of characterizing the space around a charge distribution similarity to the electric field. Notice how the potential does not depend on the charge *q* but only on the charge distribution *ρ* which generates the field ***E***. The potential is a scalar (magnitude and sign (+ or -), while electric field is a vector (magnitude and direction).
Knowing the potential, the local value of electric field ***E***(***r***) is obtained from the inverse relation

$$\vec{E}(\vec{r}) = -\frac{d}{d\vec{r}}V(\vec{r})$$

where *V*(***r***) is the potential at point ***r***.

Knowing the potential, then we can determine the potential energy of any charge that is placed in that space. If we know the field, then we can determine the force on any charge placed in that field. Electric potential, just like potential energy, is always defined relative to a reference point (zero potential). The potential difference between two points ΔV is independent of the reference point.

Example



A uniform field of magnitude E = 200 N/C is created by two oppositely charged parallel plates, as shown. What is the potential difference $\Delta V = V_C - V_A$ between points C and A? The distance from A to B is 0.3 m and the distance from B to C is 4 m.

$$\Delta V_{CA} = V_C - V_A = \Delta V_{CB} + \Delta V_{BA}$$

Since $\boldsymbol{E}$ is perpendicular to the displacement from B to C, then B and C are at the same potential. So,

$$\Delta V_{CA} = \Delta V_{BA} = -E_x(x_B - x_A) = -(200\,N/C)(0.3\,m) = -60\ V$$

The negative sign means that point C is a lower potential than point A.

$$\Delta V_{AC} = -\Delta V_{CA} = 60\ V$$

Example

If the potential difference between the positive and negative plates were 1000 V and the separation of the plates were 10 cm, what would be the magnitude of the electric field between the plates?

Since $\Delta V = -E_x \Delta x$, then

$$E_x = -\frac{\Delta V}{\Delta x} = -\frac{(-1000\,V)}{0.1\,m} = 10{,}000\ V/m\ (=10{,}000\ N/C)$$

The positive sign means that E_x points to the right, in the direction of decreasing potential.

Potential Energy of Two Point Charges

The force between two point charges is $F = k_e \frac{q_1 q_2}{r^2}$. This force can do work by pushing the charges apart (if q_1 and q_2 have the same sign) or pulling them together (if q_1 and q_2 have opposite sign). There is a potential energy associated with these two charges. The change in potential energy is the negative of the work done during the displacement. Since the force is not constant, then we must calculate this work from the area under the force versus displacement curve, or by using integral calculus.

$$\Delta PE = U_b - U_a = -W = -\int_{r_a}^{r_b} k_e \frac{q_1 q_2}{r^2} dr = k_e \frac{q_1 q_2}{r_b} - k_e \frac{q_1 q_2}{r_a}$$

Potential energy always depends on the choice of where the potential energy is assumed to be zero. For point charges, the convention is to assume that $PE = 0$ when $r = \infty$. Thus, we have

$$PE = U = k_e \frac{q_1 q_2}{r} \ .$$

Note that this equation is similar to the force formula except that PE varies inversely with r instead of r^2. Also, PE is a *scalar* (it can be + or -), whereas force is a vector.

Example

Two protons are released from rest when they are 1 cm apart. What is their speed when they have moved very far apart?

At $r = \infty$, $PE = 0$, so $\Delta PE = PE_{final} - PE_{initial} = -k_e \frac{e^2}{r}$.

$$\Delta KE = -\Delta PE$$

$$\tfrac{1}{2} m v^2 + \tfrac{1}{2} m v^2 = k_e \frac{e^2}{r} \quad (both \ \ protons \ \ are \ \ moving)$$

$$v = \sqrt{\frac{k_e e^2}{mr}} = \sqrt{\frac{(9 \, x \, 10^9)(1.6 \, x \, 10^{-19})^2}{(1.67 \, x \, 10^{-27})(0.01)}} = 3.7 \ m/s$$

Electric Potential of a Point Charge

Since $\Delta V = \frac{\Delta PE}{q}$, then the potential of a single point charge is

$$V = k_e \frac{q}{r}$$

where it is assumed that $V = 0$ when $r = \infty$.

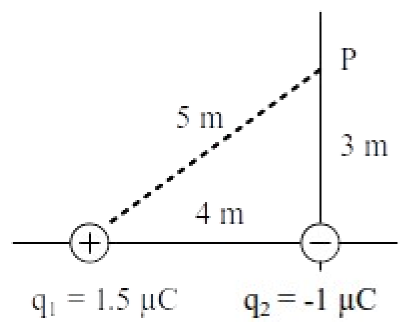
Example

What is the electric potential 50 cm from a point charge $q = 1 \times 10^{-6}$ C?

$$V = k_e \frac{q}{r} = (9 x 10^9) \frac{1 x 10^{-6}}{0.5} = 1.8 x 10^4 \ V = 18 \ kV$$

Example

What is the electric potential at point *P* in the diagram to the right?



$$V = V_1 + V_2 = k_e \frac{q_1}{r_1} + k_e \frac{q_2}{r_2}$$
$$= (9 x 10^9) \frac{(1.5 x 10^{-6})}{(5)} + (9 x 10^9) \frac{(-1 x 10^{-6})}{(3)}$$
$$= 2700 \ V - 3000 \ V = -300 \ V$$

Equipotential Surfaces

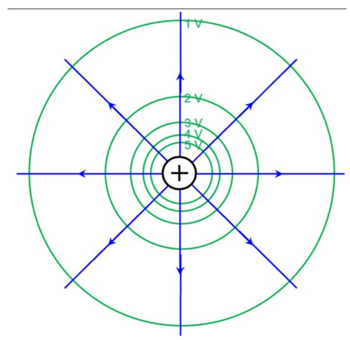
If a charge is moved perpendicular to an electric field, then no work is done and there is no change in potential energy and so no change in electric potential. Thus, any surface over which the electric field is perpendicular is at constant potential. This surface is called an “equipotential” surface (or lines in the case of two dimensional representations). At any given point the equipotential lines are perpendicular to the electric field lines.

Potential surfaces due to a point charge:

For a point charge, equipotential surfaces are spheres with the charge at the center.

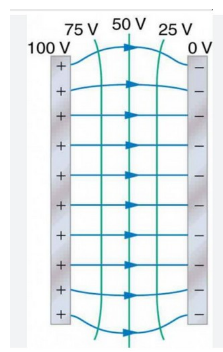
$$V(r) = k_e \frac{q}{r}$$

Every r = constant gives a equipotential surface (a green circle in two dimensions).

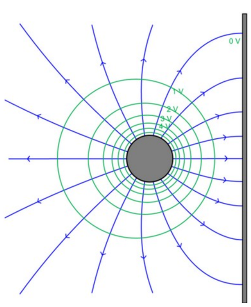


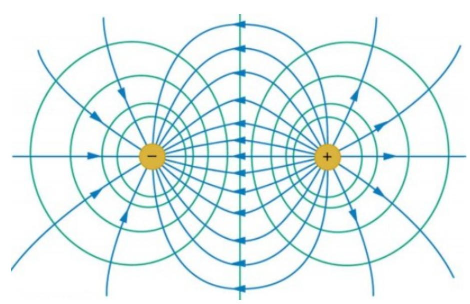
Equipotential surfaces between two parallel pates:

The equipotential surface (green lines) have value of potential which decreases from the plate at higher potential (100 V) toward the place at lower potential (0 V). The electric field is uniform.



Equipotential surfaces for a positive charge and negative plate (left below)
Equipotential surfaces for two point charges of opposite sign (right below)



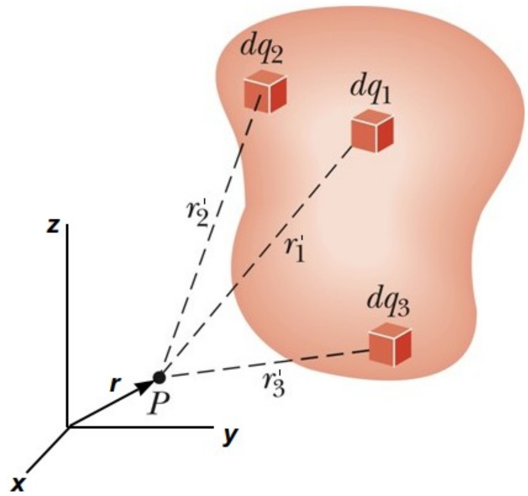


The surface of a charged conductor in equilibrium is an equipotential surface since the electric field is everywhere perpendicular to the surface. Also, the volume of a conductor is at constant potential. This is true since E = 0 everywhere inside a conductor which implies there is no change in the value of the potential $\Delta V = -E_x \Delta x = 0$ throughout the entire conductor.

Electric Potential due to a continuous charge distribution

For a continuous charge distribution ρ, the electric potential at a point P with coordinates $\boldsymbol{r}$ can be found by summer over the infinitesimal electric potentials dV_1, dV_2, …. i.e. by integration

$$V = \lim_{\Delta V \to 0} \sum_i \Delta V_i = \int dV ,$$



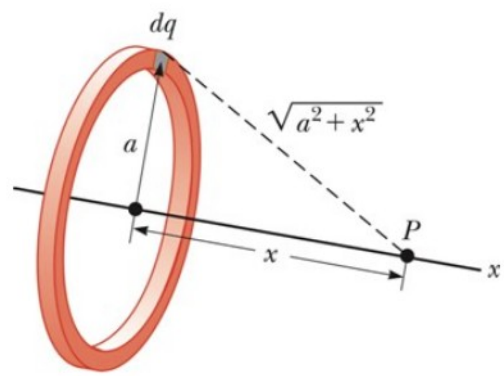
each dV_i is given by the infinitesimal point-like charges dq_i each located at distance r'_i from the point P.

$$V(r) = \int k_e \frac{dq}{r'}$$

where $\boldsymbol{r}$ is the position vector of point P respect to a chosen coordinates system. It is also assumed $V = 0$ at infinity.

Example

Find the potential on at a point P located on the x-axis perpendicular to the center of the ring of charge Q. In this example $r' = \sqrt{(a^2 + x^2)}$ and $\boldsymbol{r} = x$.



$$V(x) = k_e \int \frac{dq}{\sqrt{a^2+x^2}}$$

since all the dq are at the same distance away from P

$$V(x) = k_e \frac{1}{\sqrt{a^2+x^2}} \int dq = k_e \frac{1}{\sqrt{a^2+x^2}} Q$$

It also possible to calculate the electric field:

$$E(x) = -\frac{dV(x)}{dx} = \frac{k_e Q}{(a^2+x^2)^{3/2}}$$