Capacitance

Capacitor

A *capacitor* consists of two metal electrodes which can be given equal and opposite charges Q and $-Q$. There is an electric field between the plates which originates on Q and terminates on $-Q$. There is a potential difference between the electrodes which is proportional to Q.

$$Q = C\,\Delta V$$

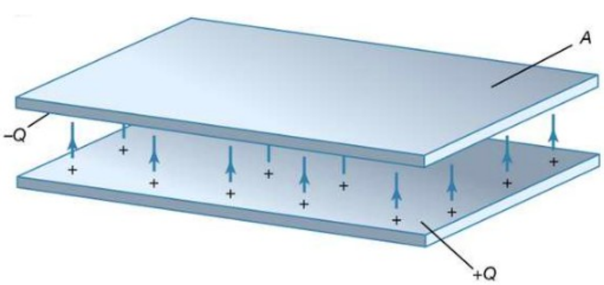
The *capacitance* is a measure of the capacity of the electrodes to hold charge for a given potential difference. As such the capacitance is operationally defined as

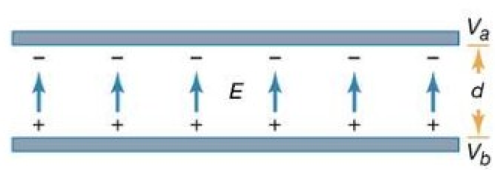
$$C = \frac{Q}{\Delta V} \qquad \text{(unit = C/V = farad = F)}$$

The capacitance is an internist propriety of any configuration of two conductors when placed next to each others. The capacitor does not need to be charged (holding a charge Q with a potential difference ΔV across the conductors) for its capacitance to exist. Capacitors come in various sizes and shapes and their capacitance depends on their physical and geometrical proprieties.

<u>Parallel plates capacitor</u>

A geometrical simple capacitor consists of two parallel metal plates. If the separation of the plates is small compared with the plate dimensions, then the electric field between the plates is nearly uniform.





The electric field between two charged plates is $E = \sigma\ \varepsilon_0$, where $\sigma = Q/A$ is the uniform charge density on each plate (with opposite sign). The potential difference between the plates is $\Delta V = V_b - V_a = Ed$, where d is the separation of the plates. The capacitance is

$$C = \frac{Q}{\Delta V} = \frac{\sigma A}{Ed} = \frac{\varepsilon_0 EA}{Ed}\ ,$$

Or

$$C = \frac{\varepsilon_0 A}{d} \quad \text{(parallel plate capacitor)}$$

The capacitance is an intrinsic propriety of the configuration of the two plates. It depends only on the separation *d* and surface area *A*.

Example

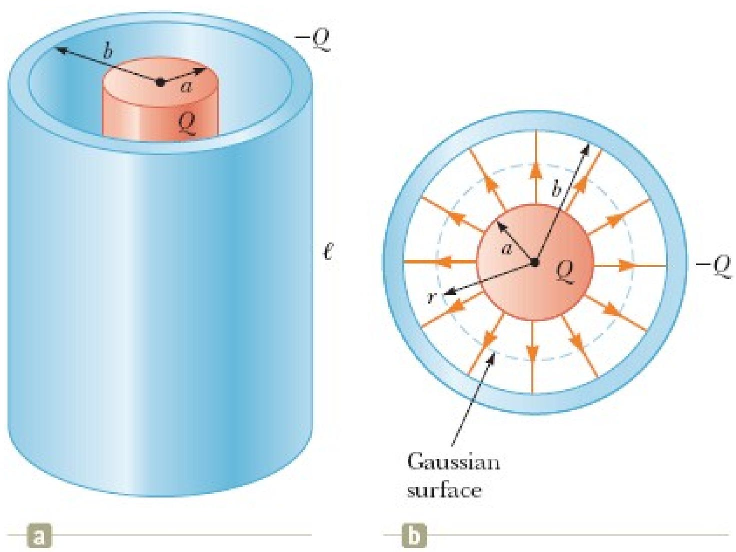
A capacitor consists of two plates 10 cm x 10 cm with a separation of 1 mm. What is the capacitance? If the capacitor is charged to a potential difference of 100 V, how much charge is on the capacitor?

$$C = \frac{\varepsilon_0 A}{d} = \frac{(8.85 x 10^{-12} C^2/N \cdot m^2)(0.1 m)^2}{(0.001 m)} = 8.85 x 10^{-11} F \ (=88.5 \ pF)$$

$$Q = C\Delta V = (8.85 x 10^{-11} F)(100 V) = 8.85 x 10^{-9} C \ (=8.85 \ nC)$$

<u>Cylindrical capacitor</u>

The capacitor consists of a metal rod of radius *a* at the center of a cylindrical shell of radius *b*. Let the rod have a charge *Q* and the shell a charge *–Q*. There is no electric field inside the rod and the charge *Q* is located on its surface. To find the capacitance first we need the expression of the electric field between the two conductors which can be found using the Gauss' law. The Gaussian surface is a cylinder with radius *r*: $a < r < b$



The flux through a cylinder of radius *r* between *a* and *b* is

$$\Phi_E = \oint E \cdot dA = E 2\pi \ rL \ ,$$

where *L* is the length of the rod and *2πrL* is the surface area of the cylinder. So,

$$E 2\pi \ rL = Q/\varepsilon_0$$

$$E(r) = \frac{Q}{2\pi \ \varepsilon_0 L \ r} = \frac{\lambda}{2\pi \ \varepsilon_0 r} \ ,$$

where $\lambda = Q/L$ is the charge per unit length.

The potential difference between the inner and outer electrodes is

$$\Delta V = V_a - V_b = -\int_b^a E\; dr = \int_a^b \frac{Q}{2\pi\ \varepsilon_0 L\ r}\, dr = \frac{Q}{2\pi\ \varepsilon_0 L} \ln\left(\frac{b}{a}\right)$$

Since the capacitance is defined as $C = Q/\Delta V$, then we have

$$C = \frac{2\pi\ \varepsilon_0 L}{\ln(b/a)} \qquad \text{(cylindrical capacitor)}$$

Example

What is the capacitance of a coaxial cable of length 1 m that has an inner wire of diameter 1 mm and an outer shell of diameter 5 mm?

$$C = \frac{2\pi\ \varepsilon_0 L}{\ln(b/a)} = \frac{2\pi\ (8.85\, x\, 10^{-12} C^2/N\cdot m^2)(1\, m)}{\ln(5)} = 3.46\, x\, 10^{-11} F = 34.6\ pF$$

<u>Spherical capacitor</u>

The capacitor consists of a metal sphere of radius *a* at the center of a spherical shell of radius *b*. Let the sphere have a charge *Q* and the shell a charge –*Q*. There is no electric field inside the sphere and the charge *Q* is located on its surface. To find the capacitance first we need the expression of the electric field between the two conductors which can be found using the Gauss' law. The Gaussian surface is a sphere with radius *r*: $a < r < b$

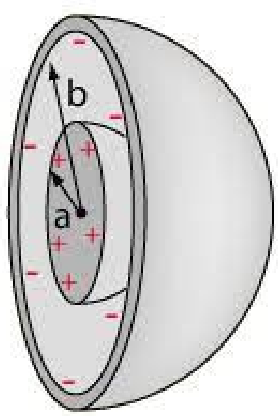
The flux through a sphere of radius *r* between *a* and *b* is

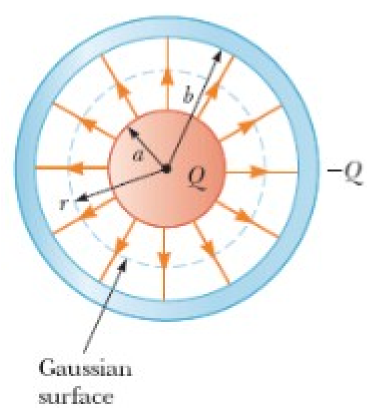
$$\Phi_E = \oint E\cdot dA = E\, 4\pi\ r^2\ ,$$

So,

$$E\, 4\pi\ r^2 = Q/\varepsilon_0$$

$$E(r) = \frac{Q}{4\pi\ \varepsilon_0 r^2}\ ,$$





The potential difference between the inner and outer electrodes is

$$\Delta V = V_a - V_b = -\int_b^a E\ dr = \int_a^b \frac{Q}{4\pi\ \varepsilon_0 r^2}\ dr = \frac{Q}{4\pi\ \varepsilon_0}\left(\frac{b-a}{ab}\right)$$

Since the capacitance is defined as $C = Q/\Delta V$, then we have

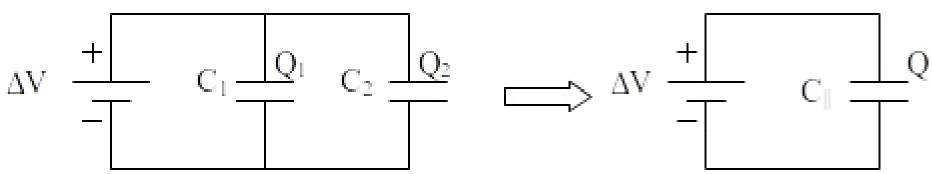
$$C = (4\pi\ \varepsilon_0)\left(\frac{ab}{b-a}\right) \qquad \text{(spherical capacitor)}$$

Combinations of Capacitors

Circuits can have multiple capacitors. In the simplest configurations, the capacitors would be either in parallel, in series, or in a combination of series and parallel.

Capacitors in parallel

In the parallel circuit, the electrical potential across the capacitors is the same and is the same as that of the potential source (battery or power supply). This is because the capacitors and potential source are all connected by conducting wires which are assumed to have no electrical resistance (thus no potential drop along the wires). The two capacitors in parallel can be replaced with a single equivalent capacitor.



The charge on the equivalent capacitor is the sum of the charges on C_1 and C_2.
Since the cables do not alter the value of the potential difference

$$\Delta V = \Delta V_1 = \Delta V_2$$

then

$$Q = Q_1 + Q_2$$
$$C_{||} \Delta V = C_1 \Delta V_1 + C_2 \Delta V_2$$

Thus,

$$C_{||} = C_1 + C_2$$

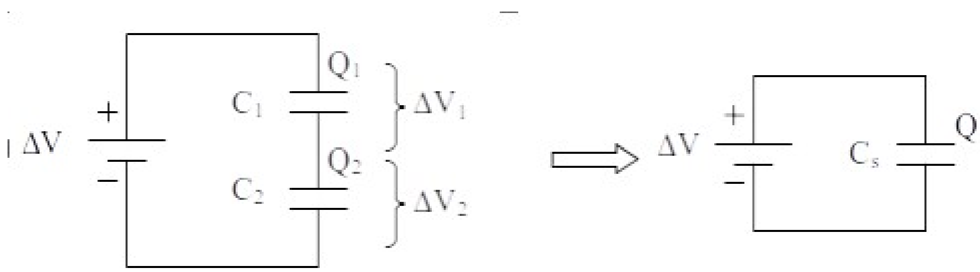
This is the meaning of equivalent capacitor.

For more than two capacitors in parallel

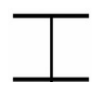
$$C_{||} = C_1 + C_2 + C_3 + \ldots$$

<u>Capacitors in series</u>

In the series circuit, the potential of the source is the sum of the potentials across C_1 and C_2. This is a consequence of conservation of energy. (A test charge moved across the source ΔV must have the same change in potential energy as a test charge moved across C_1 and then across C_2.)



The charges on each capacitor are the same. Reason: consider one single capacitor not connected to battery, the charge on each plates is zero. Connecting it to a battery (as shown in picture to the right) provides a potential difference which move the charges from on plate to the other: one plate becomes positive $+Q_1$, the other negative $-Q_2$ of equal in magnitude. Now look at the picture to the left: having two capacitors is equivalent to inserting a new neutral conductor within the plates with the shapes:



The above $+Q_1$ and the below $-Q_2$ create a separation of the charges in the conductor such that $-Q_1$ faces the $+Q_1$ above it and $+Q_2$ faces the $-Q_2$ below. All the charges have same magnitude Q.

$$Q = Q_1 = Q_2$$

then

$$\Delta V = \Delta V_1 + \Delta V_2$$
$$\frac{Q}{C_s} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2}$$

Thus,

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2}$$

Another way of writing this equation is

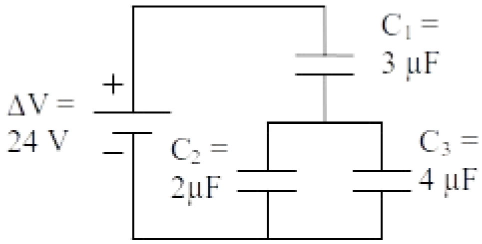
$$C_s = \frac{C_1 C_2}{C_1 + C_2}$$

For more than two capacitors in series

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$$

Example

In the circuit below, find the total equivalent capacitance, the charge on each capacitor, and the voltage across each capacitor.



Find the equivalent capacitance of C_2 and C_3.

$$C_{23} = C_2 + C_3 = 6\mu F \quad \text{parallel combination}$$

This reduces the circuit to two capacitors in series. Then, find the equivalent capacitance C of C_1 and C_{23}

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_{23}} \;\Rightarrow\; C = \frac{C_1 C_{23}}{C_1 + C_{23}} = 2\mu F$$

Use the equivalent capacitance C to find the total charge

$$Q = C \Delta V = 4.8 \times 10^{-5}\,\mathrm{C}$$

Next, let's work backward.

Since C_1 is in series with C_{23} they have equal charge $Q = Q_1 = Q_{23}$. Knowing Q_1 can find ΔV_1

$$\Delta V_1 = \frac{Q_1}{C_1} = 16\,V$$

Since C_1 is in series with C_{23} their voltage add $\Delta V = \Delta V_1 + \Delta V_{23}$. Hence $\Delta V_{23} = 8$ V
Since C_2 is in parallel with C_3 the voltage across them is the same $\Delta V_{23} = \Delta V_2 = \Delta V_3$
Knowing the the voltages we can find the charges

$$Q_2 = C_2 \Delta V_2 = 1.6\times10^{-5}\,\text{C}$$
$$Q_3 = C_3 \Delta V_3 = 3.2\times10^{-5}\,\text{C}$$

You can check that $Q = Q_1 = Q_{23} = Q_1 + Q_2$

Energy Stored in a Capacitor

The energy stored in a charged capacitor is given by

$$U = \tfrac{1}{2} Q\Delta V \ ,$$

where Q is the charge on the capacitor and ΔV is the voltage (potential) across the capacitor. If we moved a small charge q through a potential difference ΔV, the change in potential energy would be $U = q\Delta V$. The reason for the factor of ½ in the above equation is because the potential varies from 0 to ΔV as the capacitor is charged, so we must take the average to find the total work done in charging it.

Using $Q = C\Delta V$, we can alternatively write the energy stored as

$$U=\tfrac{1}{2}C(\Delta V)^2=\tfrac{1}{2}\frac{Q^2}{C}$$

Example

A 3-µF and a 6-µF capacitor are connected in parallel to a 12-V battery. Find the charge on each capacitor, the equivalent parallel capacitance, and the total energy stored.

$$Q_1=C_1\Delta V=(3\,\mu F)(12\,V)=36\ \mu C=36\,x\,10^{-6}\ C$$
$$Q_2=C_2\Delta V=(6\,\mu F)(12\,V)=72\ \mu C=72\,x\,10^{-6}\ C$$
$$C_{||}=C_1+C_2=3\,\mu F+6\,\mu F=9\ \mu F$$
$$U=\frac{1}{2}C_{||}(\Delta V)^2=\frac{1}{2}(9\,x\,10^{-6}F)(12\,V)^2=6.48\,x\,10^{-4}\ J=648\ \mu J$$

Example

The capacitors in the above example are now connected in series to the 12-V battery. Find the charge on each capacitor, the voltage across each capacitor, the equivalent parallel capacitance, and the total energy stored.

The equivalent capacitance and the charge on the equivalent capacitance are

$$C_s = \frac{C_1 C_2}{C_1 + C_2} = \frac{(3)(6)}{3+6} = 2\ \mu F$$
$$Q_s = C_s \Delta V = (2\mu F)(12\,V) = 24\ \mu C$$

We note that for series capacitors, $Q_1 = Q_2 = Q_s$. So, the voltages across the capacitors are

$$\Delta V_1 = \frac{Q_1}{C_1} = \frac{24\,\mu C}{3\,\mu F} = 8\ V$$
$$\Delta V_2 = \frac{Q_2}{C_2} = \frac{24\,\mu C}{6\,\mu F} = 4\ V$$

Note that the smaller voltage is across the larger capacitor and vice-versa. Also, note that

$\Delta V_1 + \Delta V_2 = 12\ V$, as is required.

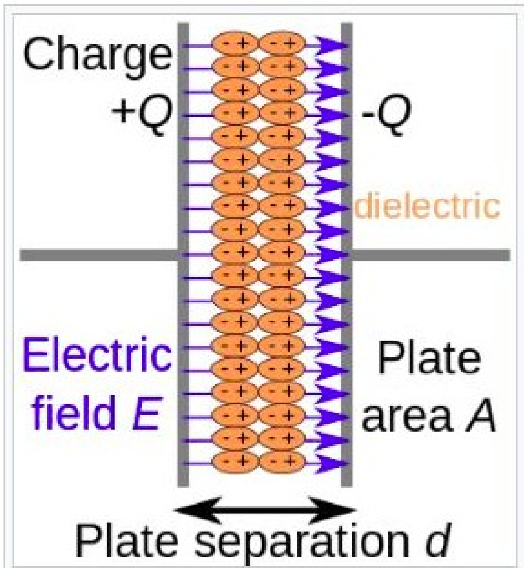
The energy stored is

$$U = \tfrac{1}{2} C_s (\Delta V)^2 = \tfrac{1}{2}(2\,\mu F)(12\,V)^2 = 144\ \mu J$$

Dielectrics

A dielectric is a insulator material which consists of polar molecules. That means that it can be polarized by an external electric field so that the net effect is that a positive charge is developed on one side of the dielectric and a negative charge on the other.

When a dielectric is placed between the plates of a charged capacitor, the polarization of the molecules reduces the electric field between the plates which implies a lower voltage across the plates. The charge Q stays the same, while the capacitance changes as well



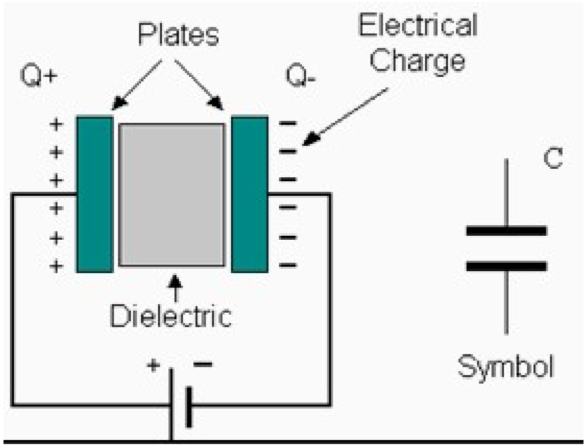
With Q_0, E_0, V_0, and C_0 indicating the values before the dielectric in place between the plates, when the dielectric is considered, we have

$$E_0 \rightarrow E = E_0/k \Rightarrow \Delta V = \Delta V_0/k$$

where k is the *dielectric constant*. For for vacuum $k = 1$ while for all others dielectric material is $k > 1$. The capacitance changes as

$$C_0 \rightarrow C = \frac{Q}{\Delta V} = \frac{Q_0}{\Delta V_0 / k} = k \frac{Q_0}{\Delta V_0} = kC_0$$

The same result is obtained if the capacitor is held at constant voltage by an external voltage source. In this case



$$\Delta V = \Delta V_0 \quad \text{and} \quad E = E_0$$

In order for **E** to stay the same the charge has to increase

$$Q_0 \rightarrow Q = kQ_0$$

which implies

$$C_0 \rightarrow C = Q / \Delta V = k\, Q_0 / \Delta V_0 = kC_0$$

The net effect of having a dielectric between the plates is to increase the capacitance by *k.*

$$C = kC_0$$

So, for a parallel plate capacitor with a dielectric:

$$C = \frac{k\, \varepsilon_0 A}{d}$$