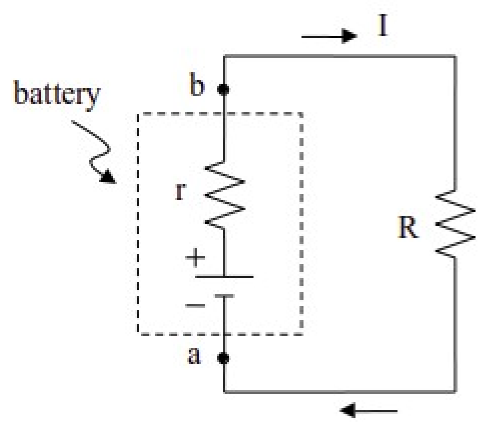
DC Circuits

This chapter deals with direct current circuits involving combinations of voltage sources (batteries or power supplies) and resistors. It also covers RC circuits in which capacitors are charged or discharged through a resistor.

Voltage sources

A voltage source in a circuit is sometimes referred to a source of emf. Emf refers to ‘electromotive force’. It is not really a force -rather it is a potential difference which can drive a current through a circuit. The most common sources would be a battery or a ‘power supply’. A power supply is an instrument which converts the AC (alternating current or alternating voltage) from the line voltage source to a DC (direct current) source.

Voltage sources are not perfect, and the voltage will vary with the amount of current that is drawn from the source. A way of modeling this is to think of a voltage source as an ideal voltage source (*emf*) in series with an internal resistor (*r*).



If the voltage source is connected to a load resistor (*R*), then it draws a current. The voltage drop across the internal resistance is *Ir*, so the total voltage that appears across the output terminals (*a* and *b*) is

$$\Delta V = emf - \Delta V_r = emf - Ir$$

This is less than the voltage that would appear if $I = 0$, which would be emf

Example

A 12-volt battery has an internal resistance of 0.1 Ω. When connected to a 1-Ω load resistor, what is the output voltage of the battery?

The output voltage of the battery must be the same as the voltage across the load. So,

$$\Delta V = \text{emf} - Ir = IR$$
$$I = \frac{emf}{r+R} = \frac{12\ V}{0.1\ \Omega + 1\ \Omega} = 10.9\ A$$
$$\Delta V = IR = (10.9\ A)(1\ \Omega) = 10.9\ V$$

How much power is dissipated in the load?

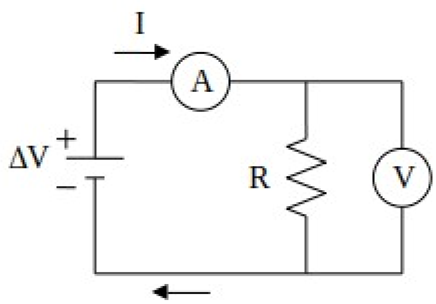
$$P = I^2 R = (10.9 A)^2 (1\ \Omega) = 119\ W$$

How much power is dissipated internally in the battery?

$$P = (10.9 A)^2 (0.1\ \Omega) = 11.9\ W$$

A simple circuit

A simple circuit with a voltage source and a single resistor is shown below. To measure the voltage across the resistor we use a voltmeter. To measure the current through the resistor we pass the current through an ammeter.



The ammeter is connected is series with the resistor because the current thought them is the same. The voltmeter is connect in parallel with the resistor because the voltage across them is the same.

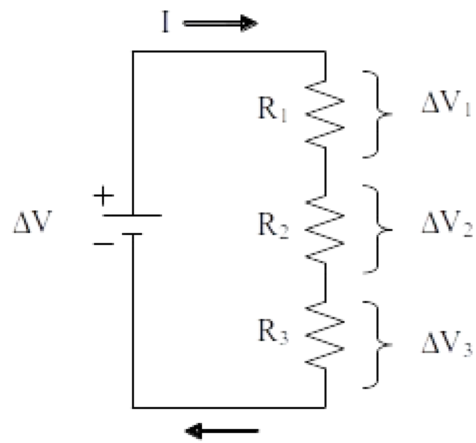
Ideally, the voltmeter should have infinite resistance so that all the current that goes through the ammeter also goes through the resistance. Also, ideally the ammeter should have zero resistance.
In practice both the voltmeter and the ammeter have finite resistance.

Resistor combinations

Series Combination

For resistors in series, the current must be the same through each resistor, and the total voltage drop across the resistors is the sum of the voltage drop across each of the resistors.

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \ldots$$
$$IR_s = I_1 R_1 + I_2 R_2 + I_3 R_3 + \ldots$$
$$I = I_1 = I_2 = I_3 = \ldots$$
$$IR_{series} = I(R_1 + R_2 + R_3 + \ldots)$$



thus,

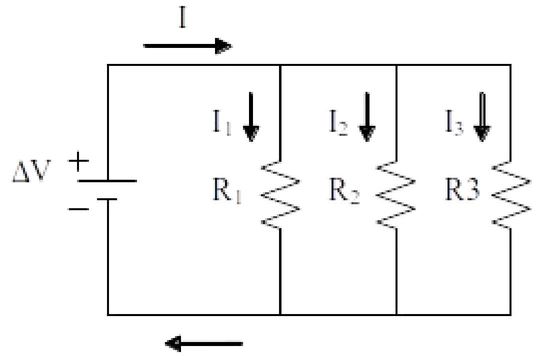
$$R_{series} = R_1 + R_2 + R_3 + ..$$

Parallel Combination

For resistors in parallel, the voltages must be the same across each resistor and the total current is the sum of the currents through each of the resistors.

$$I = I_1 + I_2 + I_3 + \ldots$$
$$\frac{\Delta V}{R_{||}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} + \frac{\Delta V_3}{R_3} + \ldots$$
$$\Delta V_1 = \Delta V_2 = \Delta V_3 = \ldots = \Delta V$$

$$\Rightarrow \quad \frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$



For two resistors in parallel, the above equation can be written as

$$R_{||} = \frac{R_1 R_2}{R_1 + R_2}.$$

Note that the rules for adding resistors in series and resistors in parallel are opposite to the rules for adding capacitors in series and in parallel.

Example

Three resistors, $R_1 = 4\ \Omega$, $R_2 = 6\ \Omega$, and $R_3 = 12$, are connected in parallel. Find the equivalent resistance.

Solution:

$$\frac{1}{R_{||}} = \frac{1}{4}+\frac{1}{6}+\frac{1}{12}=\frac{6}{12}=\frac{1}{2} \Rightarrow R_{||} = 2\ \Omega$$

Note that $R_{||}$ is smaller than any of the individual resistances.

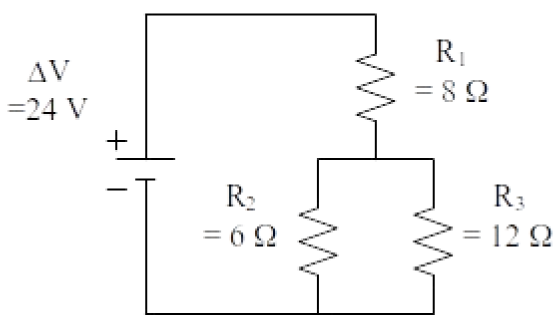
If connected in series, then the equivalent resistance is

$$R_{series}=4+6+12=22\ \Omega$$

R_{series} is larger than any of the individual resistances.

Example

Find the current through and the voltage drop across each resistor in the circuit below.



We first find the total equivalent resistance. R_2 and R_3 are in parallel, and this parallel combination is in series with R_1. Thus,

$$R_{23} = \frac{R_2 R_3}{R_2+R_3}=\frac{(6)(12)}{6+12} = 4\ \Omega$$
$$R_{eq} = R_1+R_{23}=8+4=12\ \Omega$$

The total current, which is also the current through R_1, is

$$I = \frac{\Delta V}{R_{eq}} = \frac{24\ V}{12\ \Omega} = 2\ A \quad (= I_1)$$

Then,

$$\Delta V_1 = I_1 R_1 = (2\ A)(8\ \Omega) = 16\ V$$
$$\Delta V_2 = \Delta V_3 = \Delta V_{23} = IR_{23} = (2\ A)(4\ \Omega) = 8\ V$$
$$I_2 = \frac{\Delta V_2}{R_2} = \frac{8\ V}{6\ \Omega} = 1.33\ A,\ \ I_3 = \frac{\Delta V_3}{R_3} = \frac{8\ V}{12\ \Omega} = 0.67\ A$$

Note that, as required,

$$I_1 = I_2 + I_3$$
$$\Delta V_1 + \Delta V_{23} = \Delta V$$

Kirchhoff's Rules

In the previous problem we saw that the currents at the junction of the resistors added and that the voltages across the resistors added to give the supply voltage. These are very general results and apply to all circuits. They are referred to as *Kirchhoff's* rules:

1. $\sum_i I_i = 0$ The sum of the currents entering a junction must be the sum of the currents leaving a junction. (*node rule*).

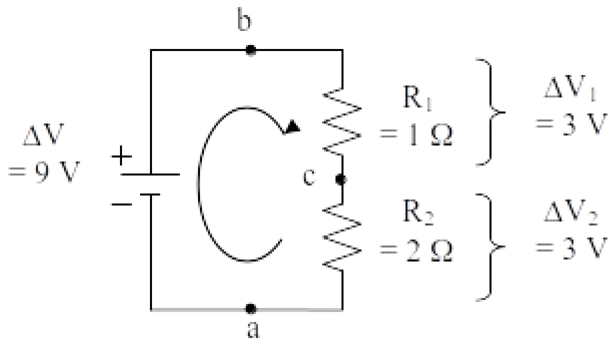
2. $\sum_i \Delta V_i = 0$ The sum of the potential changes around a closed loop must add to zero. (*loop rule*).

The first rule is based on conservation of charge and the second rule is based on conservation of energy.

In interpreting and applying the loop rule, we have to be careful about signs. ΔV can be positive or negative, depending on whether V increases or decreases while traversing the loop across a circuit element.

Example

Let's apply the loop rule to the simple series circuit below.



We must first pick a direction (clockwise or counter-clockwise) about which to traverse the loop in summing the potential changes. Either direction will work. As indicated above, we choose the clockwise direction. Starting at point a, applying the loop rule gives

$$\Delta V - IR_1 - IR_2 = 0$$

We can solve this for *I* and get *I* = 9V/3Ω = 3 A. Then,

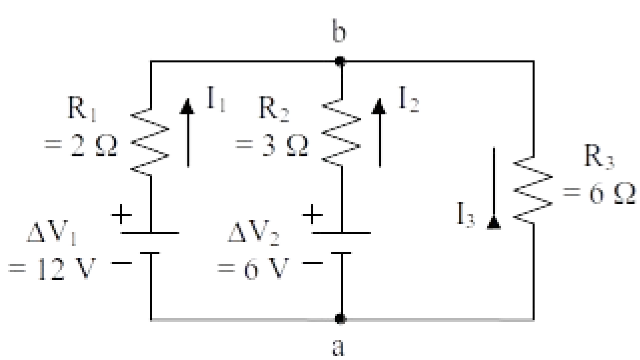
$$12V - (3A)(1\Omega) - (3A)(2\Omega) = 12V - 3V - 6V = 0$$

In going from point *a* to point *b*, the potential increases. In going from *b* to *c* the potential decreases. (Current flows in the direction of decreasing potential.) And in going from *c* back to *a*, the potential also decreases. If we had summed the potential changes going counter-clockwise, all the signs in the above equation would have been reversed, but the results would have been the same.

The Kirchhoff's Rules are used to solve general circuits containing more than one power supplies or when the combination of the electric components cannot be simplified as series and/or paralleled combination.

Example

Use Kirchhoff's rules to determine the current in each part of the circuit below.



Applying the loop rule in the clockwise direction to the left loop, starting from point a, we have

$$\Delta V_1 - I_1 R_1 + I_2 R_2 - \Delta V_2 = 0$$
$$12V - I_1(2\Omega) + I_2(3\Omega) - 6V = 0$$

Dropping the units and simplifying, the above equation is

$$\boxed{2I_1 - 3I_2 = 6} \quad \text{Eq. (1)}$$

Note that the sign of the potential change when going across a resistor depends on whether we are going in the direction of the current (-) or opposite to the current (+).

For the right loop, starting from a and going clockwise, we have

$$\Delta V_2 - I_2 R_2 - I_3 R_3 = 0$$
$$6V - I_2(3\Omega) - I_3(6\Omega) = 0$$

Again, dropping units and simplifying, we have

$$\boxed{I_2 + 2I_3 = 2} \quad \text{Eq. (2)}$$

The sum rule, applied to either junction a or b, gives

$$\boxed{I_1 + I_2 = I_3} \quad \text{Eq. (3)}$$

The three boxed equations involve three unknowns, I_1, I_2, and I_3. We can solve these equations to get (do the algebra)

$$I_1 = 2\ A \qquad I_2 = -0.67\ A \qquad I_3 = 1.33\ A$$

Note that the sign of I_2 is <u>negative</u>. This means that the choice of the direction of I_2 indicated in the diagram (up) was reversed. The current is really down.

Nowadays complex circuits containing multi loops and several currents are solved using computer software.

RC Circuits

A RC circuit consists of a resistor and a capacitor. We will consider only the series combination of them.

Charging a capacitor

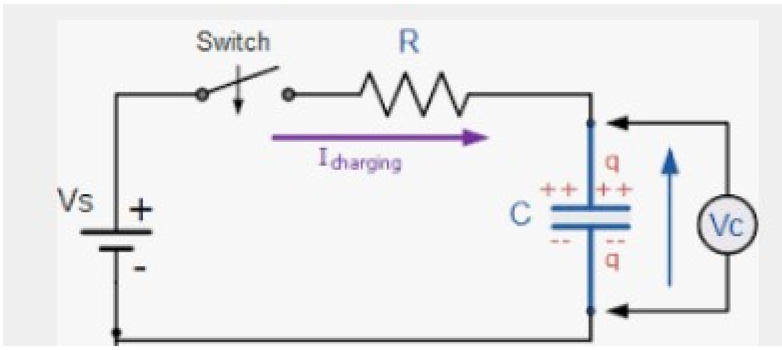
Initial conditions: Power supply provides the constant potential difference $V_S = \mathcal{E}$
The charge on the capacitor is $Q = 0$.

As the process of charging starts (for example by closing a switch) the *E* moves positive charges from one plates to the others resulting in the capacitor with charge *Q*. To study how the value of the charge as a function of time we use the Kirchhoff's loop equation

$$V_S - V_R - V_C = 0$$

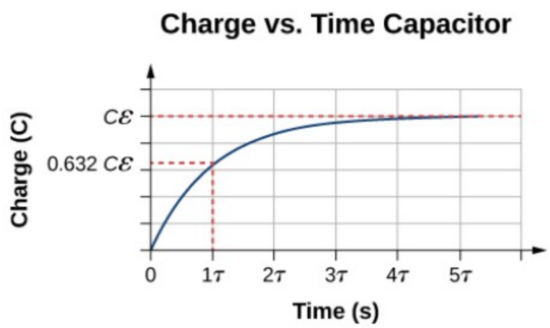
or

$$V_S - \frac{dQ}{dt}R - \frac{Q}{C} = 0$$



which is a differential equation in *Q(t)*. The equation can be solved by separation of variable leading to

$$Q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right) = Q_{MAX}\left(1 - e^{-\frac{t}{RC}}\right)$$



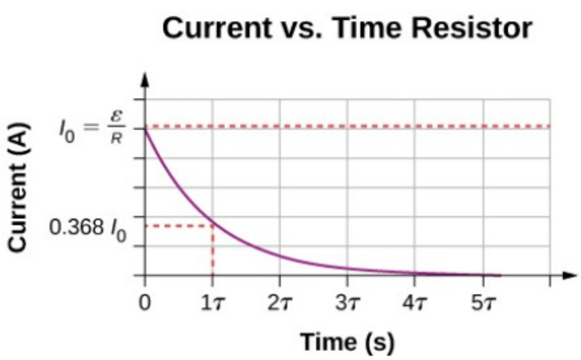
The charge increase the a maximum value

$$Q_{MAX} = \mathcal{E}C$$

The time $t = RC = \tau$ is the *time constant* of the circuit. For $t = \tau$, about 63% of Q_{MAX} is present already on the capacitor.

The current across the resistor is

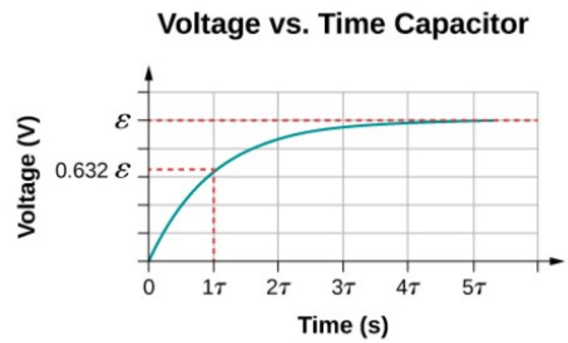
$$I(t) = \frac{dQ(t)}{dt} = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}} = I_{MAX}\,e^{-\frac{t}{RC}}$$



The current through the resistor is greatest when the charging starts and decreases exponentially in time and approaches zero as the capacitor voltage reaches the battery voltage.

The voltage across the capacitor V_C is

$$V_C = \frac{Q}{C} = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)$$



We can evaluate V_C at some relevant times:

$$t = 0,\ \ V_C = \mathcal{E}\left(1 - e^{0}\right) = \mathcal{E}\left(1 - 1\right) = 0\ .$$

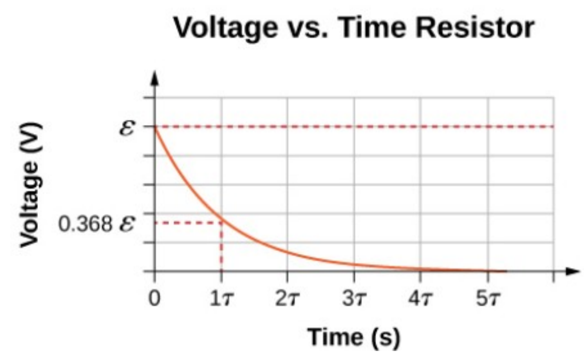
$$t = \text{infinity},\ \ V_C = \mathcal{E}\left(1 - e^{-\infty}\right) = \mathcal{E}\left(1 - 0\right) = \mathcal{E}$$

$$t = RC,\ \ V_C = \mathcal{E}\left(1 - e^{-1}\right) = \mathcal{E}\left(1 - 0.368\ldots\right) \approx 0.632\,\mathcal{E}$$

RC has units of time and is a measure of how long it takes for the capacitor to charge through the resistor. (Actually, it takes an infinite time to fully charge; however, it charges up to nearly 2/3 its final value (0.632…) in the time *RC*).

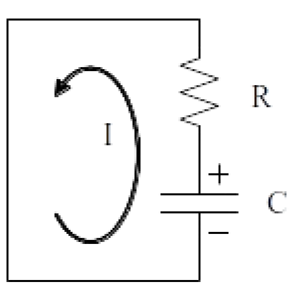
The Voltage across the resistor V_R is

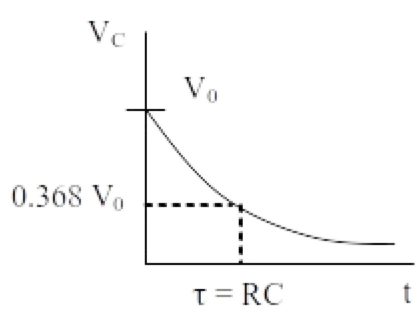
$$V_R = \mathcal{E} e^{-\frac{t}{RC}}$$



<u>Discharging a capacitor</u>

When a capacitor is discharged through a resistor, then its voltage and charge decrease exponentially in time.





$$V_C = V_0 e^{-t/RC}\ , \qquad Q = CV_C = CV_0 e^{-t/RC}$$

The discharge current is opposite to the charge current and its magnitude also decreases exponentially.

Evaluating V_C at some relevant times, we have

$t = 0$, $V_C = V_0 e^0 = V_0$.

$t =$ infinity, $V_C = V_0 e^{-\infty} = 0$

$t =$ RC, $V_C = V_0 e^{-1} = 0.368 V_0$

Example

A 10 μF capacitor is charged by a 24 volt battery through a 400 kΩ resistor. After 1 s, what is the voltage across the capacitor, the charge on the capacitor, the current to the capacitor, and the voltage across the resistor?

The time constant is $\tau = RC = (400 \times 10^3\ \Omega)(10 \times 10^{-6}\ F) = 4$ s

$$V_C = \mathcal{E}(1 - e^{-t/RC}) = 24\,V(1 - e^{-1/4}) = 24\,V(1 - 0.779) = 5.3\ V$$
$$Q = CV_C = (10\,x\,10^{-6}\,F)(5.3\,V) = 5.3\,x\,10^{-5}\,C$$
$$I = \frac{\mathcal{E}}{R} e^{-t/RC} = \frac{24\,V}{4\,x\,10^5\,\Omega} e^{-1/4} = 4.57\,x\,10^{-5}\ A$$
$$V_R = IR = (4.57\,x\,10^{-5}\,A)(4\,x\,10^5\,\Omega) = 18.7\ V$$

Note that $V_C + V_R = \mathcal{E}$.

Example

A 2 μF-capacitor is charged to 50 volts and then discharged through a 1 MΩ resistor. Find the time for the capacitor voltage to drop to 25 volts.

$$V_C = V_0 e^{-t/RC}$$
$$25\,V = 50\,Ve^{-t/RC}$$
$$e^{-t/RC} = 25/50 = 0.5$$
$$\ln(e^{-t/RC}) = -\frac{t}{RC} = \ln(0.5) = -\ln(2)$$
$$t = RC\ln(2) = (1\,x\,10^6\,\Omega)(2\,x\,10^{-6}\,F)\ln(2) = 1.39\ s$$