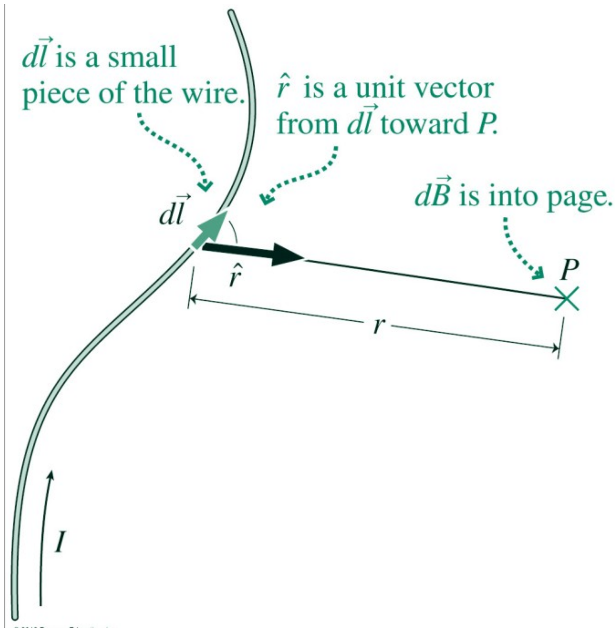
Magnetic Field Sources

In 1820 Danish physicist Hans Christian Ørsted noted that when a compass is placed next to a wire carrying current, its needle would turn. By further investigations he discovered the proprieties of the **B** field produced by a current in a straight wire.

Biot-Savart Law

Given the experiment fact that a current is a source of a magnetic field, the next step was to formulate the general equation in able to provide the **B** field due to a *steady* current *I* ($dI/dt = 0$) moving in a wire of any shape. This equation is the *Biot-Savart* Law.

$$\vec{B} = \int d\vec{B} = \\ = \frac{\mu_0}{4\pi} \int \frac{I\, d\vec{l} \times \hat{r}}{r^2}$$

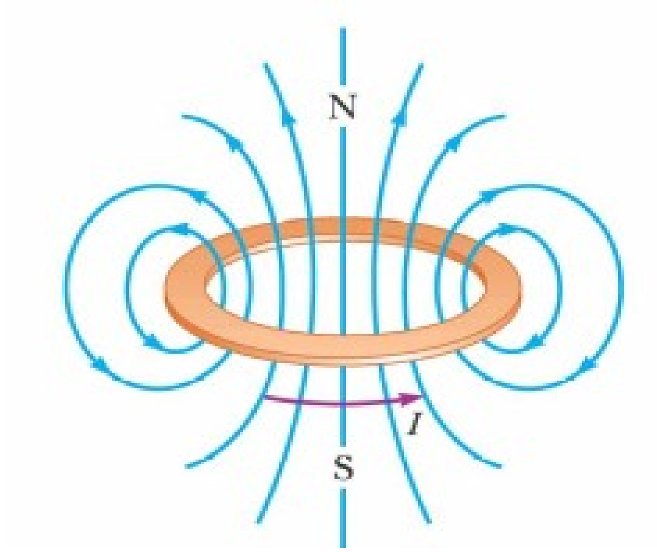


where:
I is the current.
dl is the infinitesimal line element.
$\hat{r}$ is the unit vector pointing toward point *P*.
r is the distance between *P* and d**l**.
$\mu_0 = 4\pi \times 10^{-7}$ N/A^2 (Newton// Ampere^2) is a constant called the permeability of free space (or magnetic constant). It plays for magnetism the analog role of ϵ_0 for electricity.

The integration is over the entire wire, × indicates the cross product of the vector ***dl*** with the unit vector $\hat{r}$ also referred as *versor.*

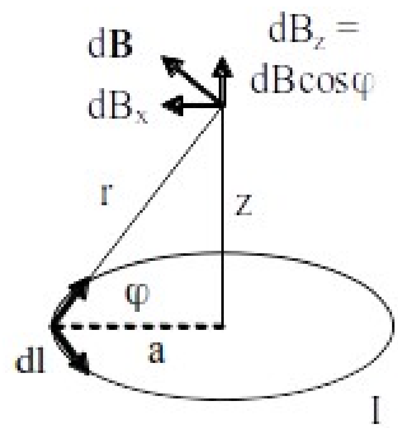
Magnetic field due to a circular current loop

The field lines due to a current loop are shown qualitatively to the right. The field passes through the loop in a direction given by still another right hand rule. If your fingers circulate around the loop in the direction of the current, then the field passes through the loop in the direction of your thumb. The field along the axis of the loop can be calculated using the Biot-Savart law.



The magnitude of the field due to the element of the wire *dl* is given by

$$dB=\frac{\mu_0 I}{4\pi}\frac{dl\ \sin 90^0}{r^2}=\frac{\mu_0 I}{4\pi}\frac{dl}{(z^2+a^2)},$$



where *z* is the distance from the center of the loop and a is the radius of the loop. The field makes an angle φ with the *z*-axis. The *x*-components of the field due to all the elements of the loop cancel and the *z*-components add. So the total field along the *z*-axis is

$$B=\int dB\cos\phi=\frac{\mu_0 I}{4\pi}\int\frac{dl\cos\phi}{(z^2+a^2)}=\frac{\mu_0 I}{4\pi}\frac{a}{(z^2+a^2)^{3/2}}\int dl$$

Since $\int dl = 2\pi\ a$, so

$$B(z)=\frac{\mu_0 I\ a^2}{2(z^2+a^2)^{3/2}}$$

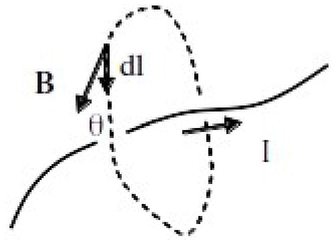
In the center of of the loop $z = 0$ this expression reduces to

$$B=\frac{\mu_0 I}{2\ a}$$

Ampere's Law

The Ampere Law is another formula which provides the relation between the **B** field and the source current *I*. It contains the same information of the Biot-Savart Law but is written in a different mathematical format.
Ampere's law relates the *circulation* of the magnetic field **B** to the current that produces the **B** field. It is given by

$$\oint \vec{B}\cdot d\vec{l} \;=\; \mu_0 I_{ENC}$$



The circulation is any geometrical closed line: the dash loop.
The integration is over the infinitesimal line element ***dl.***
I_{ENC} is the net current enclosed by the circulation.

Note: the ***dl*** in the Ampere law is significantly different from the ***dl*** in the Biot-Savart law. In the latter ***dl*** is tangent to a physical wire, while in the Ampere law ***dl*** is tangent to the circulation which is not physical.

<u>RHS (Right Hand Side) of Ampere Law</u>

If $I_{ENC} = 0$ does mean then **B** = 0? No.
It simply meas that performing the integration (= summing the vales of **B)** along a close loop gives as result the number 0. The **B** filed is non-zero in the surroundings.

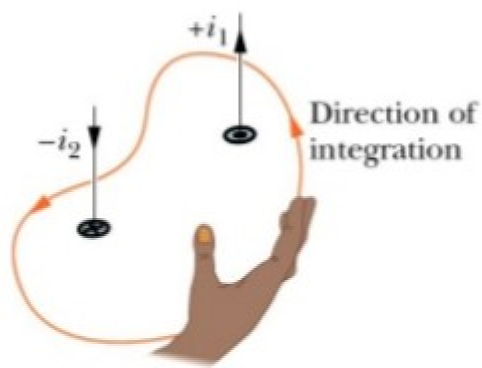
If $I_{ENC} = 0$ does mean there are no currents? No.
I_{ENC} is the net current within the loop. There is no restriction on the number of currents, some of which might be positive (in one direction), others can be negative (opposite direction) and they might cancel out.

The sign convention for the current depends on the direction of the integration, as follow

$I_{ENC} = \sum I_i$ where the current I_i has sign

positive: if it agrees with the right hand rule.
negative: if it disagrees with the right hand rule.

Example



Find the enclosed current corresponding to the closed loop in the figure on the right. The two currents have equal magnitude.

$$I_{ENC} = +i_1 + (-i_2) = i_1 - i_2 = 0$$

which does not imply **B** = 0.

LHS (Left Hand Side) of Ampere Law

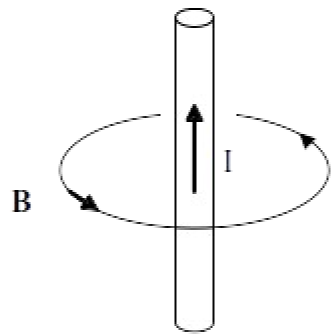
If the problem has sufficient symmetry, Ampere's law can be used to calculate **B**. The trick is similar to the Gauss Law: the result of the integration does not depend on the path of the circulation, so we can choose a circulation which matches the geometry such that **B** is constant along the path and it can be pulled out of the integration.

Magnetic field due to a long straight current-carrying wire

If the loop is chosen to be a circle of radius *r* with the wire through the center, then by symmetry **B** has the same magnitude at all points on the circle. Also, from the Biot-Savart Law we can conclude that the direction of **B** is everywhere tangent to the circle.

$$\oint B \cdot dl = B \oint dl = B(2\pi r) = \mu_0 I$$

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

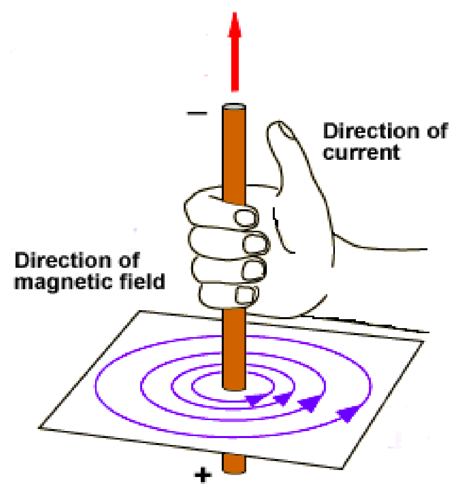


where *r* is the distance from the center of the wire. The field circulates about the wire and decreases in magnitude with distance from the wire.

The direction of the magnetic field is determined by a right hand rule:

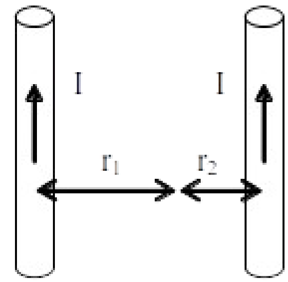
Grab the wire with your hand with you thumb pointing in the direction of the current. Then your fingers circulate around the wire in the direction of the magnetic field.

Note: this RHR is different from of the Lorentz force (a cross product which relates three quantities). In this case there are only the two physical quantities *I* and **B**.



Example

Two parallel wires 20 cm each carry a current of 5 A. Find the magnitude and direction of the magnetic field at a point between the wires 15 cm from one wire and 5 cm from the other.

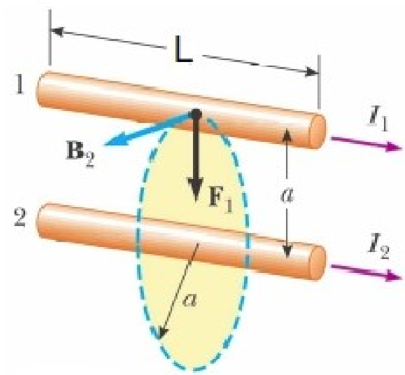


At the point in question, the field due to the left wire points into the page and the field due to the right wire points out of the page. Letting out of the page be the positive direction (this is arbitrary), we have

$$B = -B_1 + B_2 = \frac{\mu_0 I}{2\pi r_1} - \frac{\mu_0 I}{2\pi r_2} = -\frac{(4\pi\ x10^{-7} T\cdot m/A)(5A)}{2\pi(0.15m)} + \frac{(4\pi\ x10^{-7} T\cdot m/A)(5A)}{2\pi(0.05m)}$$
$$= -6.67x10^{-6} T + 2x10^{-5} T = 1.33x10^{-5} T \ \ (out\ \ of\ \ page)$$

Force between two current carrying wires

Two parallel current-carrying wires will experience an attractive or a repulsive force, depending on whether the currents are in the same direction (attractive) or in opposite directions (repulsive). Each wire sees the field created by the current in the other wire. By combining the formula for the field generated by one wire and the force on the other wire due to this field, we get



$$F = \frac{\mu_0 I_1 I_2 L}{2\pi a}$$

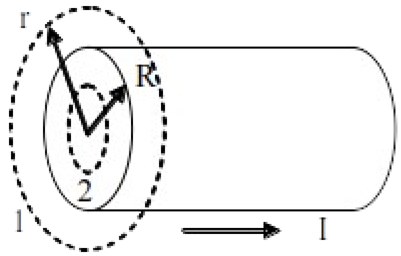
where *a* is the separation of the wires and *L* is the length of the wires.

Example

For the two wires in the previous example, the force per unit length is

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi a} = \frac{(4\pi\ x10^{-7} T\cdot m/A)(5A)^2}{2\pi(0.2m)} = 2.5x10^{-5} N/m$$

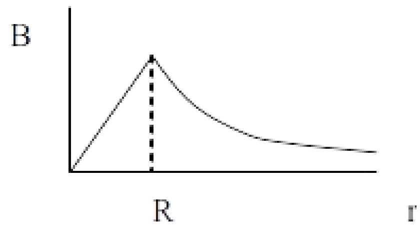
Magnetic field due to a solid wire



A wire with radius *R* that carries a uniform current *I*.

If we apply Ampere’s law to a loop of radius r outside the wire (loop 1) then we get the previous expression for the magnetic field. However, inside the wire (loop 2), the current through the loop is a fraction of the total current. This current is

$$I' = JA = \left(\frac{I}{\pi R^2}\right)\pi r^2 = \frac{r^2}{R^2} I$$



So,

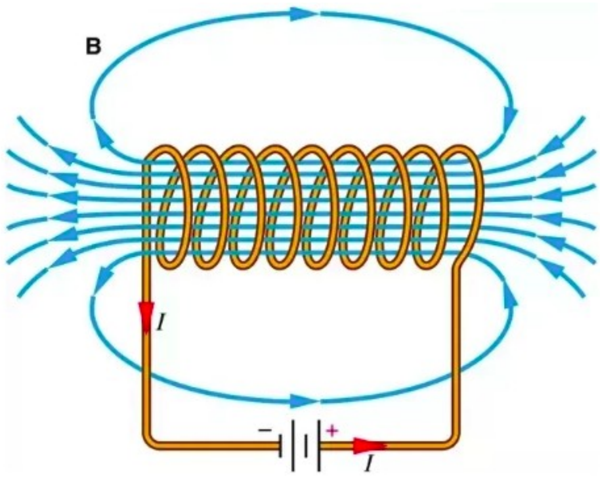
$$\oint B \cdot ds = B\, 2\pi r = \mu_0 I' = \mu_0 \frac{r^2}{R^2} I$$

$$B = \frac{\mu_0 I r}{2\pi R^2}, \quad r < R$$

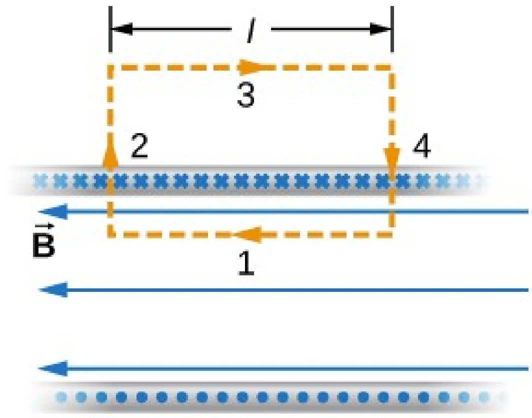
So, the magnetic field at the center of the wire is zero and it is a maximum on the surface of the wire.

Magnetic field of a solenoid

A solenoid is a cylindrical coil of wire. The magnetic field due to a solenoidal current is shown below. In the approximation of a long solenoid compared to its diameter, the field on the outside is similar to a bar magnet and negligibly weak.



To calculate **B** inside, we pick a rectangular loop as shown to the right, part of which is inside and part of which is outside. The only contribution to the integral in Ampere's law is from the path 1. Paths 2 and 4 are perpendicular to **B** (dot product is zero) and outside the field is weak. If the loop has length *l* and encloses *N* turns, then we have



$$\oint \vec{B}\cdot d\vec{l} = B\ l = \mu_0 NI$$

or more generally for solenoid of total length $l = L$

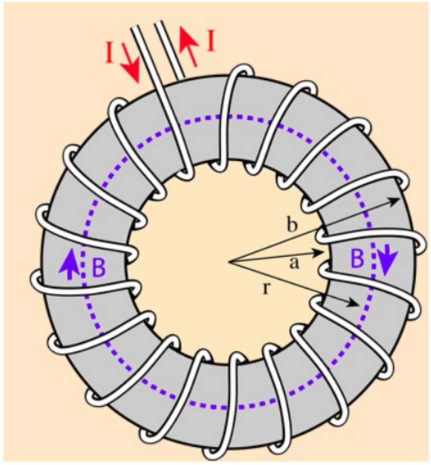
$$B = \mu_0 nI\ ,$$

where $n = N/L$ is the number of turns per unit length.

Note: the **B** field inside is nearly uniform. It is the analog case of the **E** field between two plates, therefore is a very simple field to deal with mathematically.

<u>Magnetic Field of a Toroid</u>

A toroid is a coil wound around a doughnut shaped form. If the windings of the toroid are close together, then the field is contained entirely within the toroid windings and is in a circular direction.



Applying Ampere's law to a circular path within the toroid (the blue dashed line) we obtain

$$\oint \vec{B}\cdot d\vec{l} = B\,L = \mu_0 NI$$

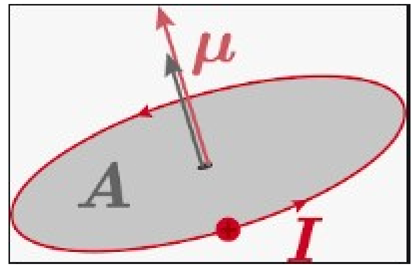
where we used the approximation that if the diameter of the toroid is large compared with the thickness $2r >> (b - a)$, then B is nearly constant from inside to outside ($a < r < b$) and we have the expression of B

$$B(r) = \frac{\mu_0 NI}{2\ \pi\ r}$$

where $L = 2\,\pi\, r$. The toroid is a useful device used in everything from tape heads to tokamaks (nuclear fusion reactor).

Microscopic Interpretation of Magnetism

In order to explain the origin of the magnetic field due to a ferromagnetic material is necessary to explore the atomic structure of the material. At the atomic level each electron spinning around its nucleus generates an electronic current and therefore it is a source of a **B** field (Ampere Law). Each atom is a like a very weak magnet.



The magnetic moment of each atom is

$\vec{\mu} = IA\,\hat{n}$ where **n** is the normal to A

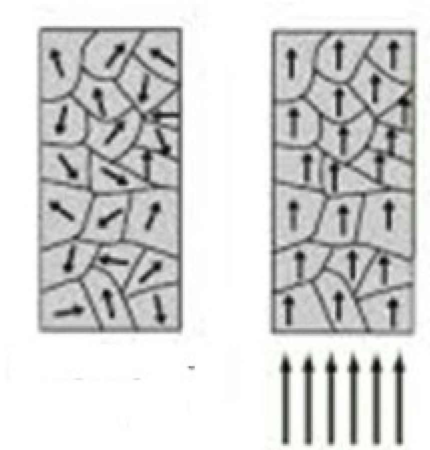
and the **B** field produced by the atom is in the same direction of **n**. In a ferromagnetic material, example a rod of iron, the magnetic moments of its atoms point in random directions.

If the iron rod is placed in a region where there is an external magnetic field $\mathbf{B}_{EXT}$, then there is torque $\vec{\tau} = \vec{\mu} \times \vec{B}_{EXT}$ acting on each atoms which tends to align them with the external field $\mathbf{B}_{EXT}$.

As consequence each **B** field of the atoms (the black arrow in the figure to the right) will point in the same direction. When $\mathbf{B}_{EXT}$ is removed, the atoms will maintain their new orientation and the iron rod is magnetized.

Due mainly to heat, the atoms will move back to random orientation and the iron rod looses its propriety of being a magnet.

Permanent magnets, example a rod of Alnico, has different atomic proprieties such that once its atoms get aligned by $\mathbf{B}_{EXT}$ they preserve their new configuration over time, allowing them to become permanent magnets.



No $\mathbf{B}_{EXT}$ Yes $\mathbf{B}_{EXT}$

The Sources of the Electric and Magnetic fields

From the microscopic interpretation of magnetism it follows that the magnetic field due to a magnet is simply a consequence of the Ampere Law, therefore magnets are not considered intrinsic source of **B**.

The following table summarize the possible sources of **E** and **B** we have learned so far

sources of **E**	*Q the charge*	
sources of **B**	*I the current*	

The column on the far right is left blank intentionally: there are actually two more ways to generate **E** *(*Faraday Law*)* and **B** (Ampere-Maxwell Law). We will learn these sources in the following chapters.

Since the current is due to charges in motion and since motion is relative to the observer, then different observes detect **E** and **B** differently.

Example

A plastic rod with charge Q is placed on a desk and does not move. There are two observers: Alice and Bob. Alice sits on a chair in front of the desk, Bob is walking in the room. If each observer has a device which measures both **E** and **B** what do they measure?

Q is at rest respect to Alice, therefore she measures the **E** due to Q.
Q is in motion respect to Bob, therefore for him a electric current exists. Bob measure also a **B** field due to the current (Ampere law). Bob also measures a **E'** field due to Q. This **E'** is different from **E** since, loosely speaking, some of the **E** observed by Alice became the **B** and **E'** fields measured by Bob.

From this considerations we deduce that **E** and **B** are just two aspects of the more general concept: the electromagnetic field. We will not study the electromagnetic field as a single physical quantity in PH106 since more advance mathematics (tensor calculus) is necessary.