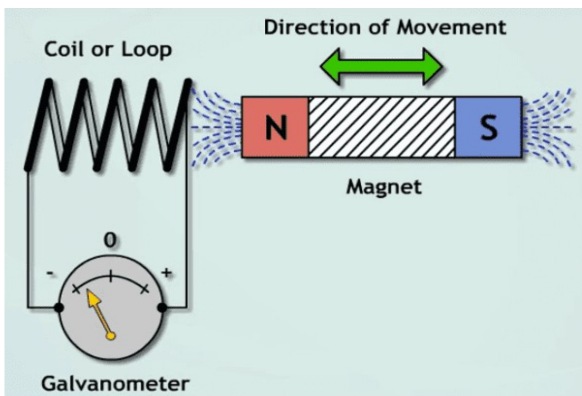
Induced Voltages and Inductance

In previous sections we have seen how electric fields are produced by charges and magnetic fields are produced by currents. Electricity and magnetism were viewed independently, except for the fact that electric fields drive currents which can produce magnetic fields and forces act on charges moving in a magnetic field. However, it turns out that electric and magnetic fields are intimately related. We can generate an electric field and a current with a time-varying magnetic field. Later we will also see how a changing electric field can produce a magnetic field.

Induced EMF

A magnet moved near a coil of wire induces a change in the potential around the coil (an induced emf) which generates a current (induced current) in the coil. The induced emf and current depend on the rate of the change of the magnetic flux through the coil.



Faraday's Law

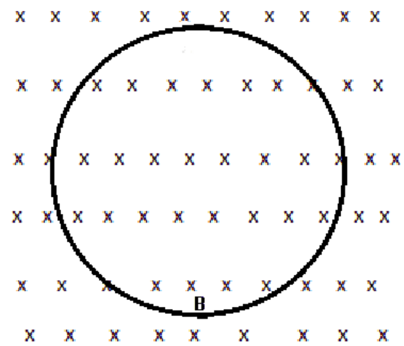
The induced emf in a loop depends on the rate at which the magnetic flux changes through the loop. Faraday's law gives the induced $\mathcal{E}_D$ as

$$\mathcal{E}_D = -N\frac{d\,\Phi\left(\vec{B}(t)\right)}{dt}$$

where N is the number of turns of wire in the coil and dt is the time during which the flux changes. The induced emf is originated by the change in time of the **B** field in each loop. When the **B** field does not vary, then there is no induced emf. Since motion is relative, the induced emf is generated by either moving the magnet or moving the coil.

Example

A circular wire is placed in a region where there is a **B** field. If the field changes in time, **B** = **B**(t) an induced emf is present in the wire.



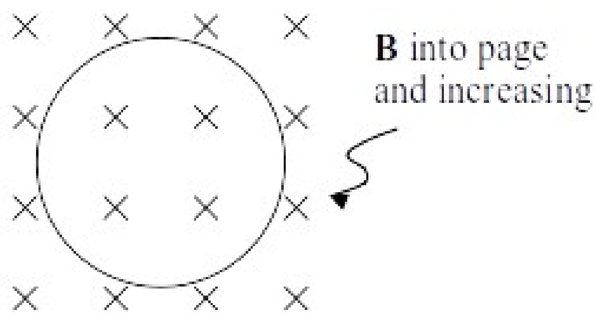
When the loop is a wire of resistance R which obeys the Ohm law, the induce current I_D is

$$I_D = \frac{1}{R}\mathcal{E}_D$$

The fundamental importance of Faraday Law is that it is the third of Maxwell equations.

Example

A magnetic field points into the page, in which lies a circular coil of wire of radius 2 cm. If the field is increased from 0.5 T to 0.6 T in a time of 0.05 s. What is the average induced emf in the loop during this time?



$$\Delta\Phi_B = \Phi_{B,f} - \Phi_{B,i} = B_f A - B_i A = (\Delta B) A = (0.1\,T)\pi\,(0.02\,m)^2 = 1.26\,x\,10^{-4}\,Wb$$

$$|emf| = \frac{\Delta\Phi_B}{\Delta t} = \frac{1.26\,x\,10^{-4}\,Wb}{0.05\,s} = 2.51\,x\,10^{-3}\,V$$

What is the induced current? For example if the resistance is R = 0.02 , then the induced current is

$$I_D = \frac{|emf|}{R} = \frac{2.51\text{x}10^{-3}\,V}{0.02\,\Omega} = 0.126\,A$$

Note:

This is the ‘integral form’ of Faraday Law since the total flux (integration over d**A**) is considered. When expressed in this form it also includes the cases of an emf generated by the change of the area or the angle in the flux, i.e. the motional emf due to the Lorentz force, see next paragraph on Flux rule. Even if mathematically the Faraday law and the Flux rule looks identical, the physics is significantly different: the term “Faraday Law” should refer only to an induced emf generated by a time varying **B** field. This is not always the case in the literature, so be careful. Instead when the Faraday law is expressed in differential form (using vector calculus, see more advanced courses) this ambiguity does not arise.

Lenz's Law

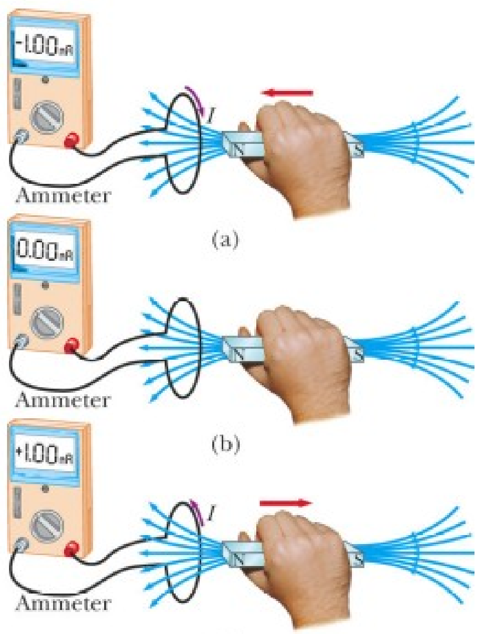
The minus sign in Faraday's law is symbolic and is refereed as the Lenz's Law. Lenz's law determines the direction of the induced emf and therefore the direction of the induced current. The induced current I_D, because the Ampere Law, generates its own induced field $\mathbf{B}_D$. This field is different from the field $\mathbf{B}$ of Faraday Law. To find the direction of I_D first we need to determine the direction of $\mathbf{B}_D$. The Lenz's law tells us how:
the direction of $\mathbf{B}_D$ *opposes the change* in the original flux.

It is important to note that it is not the magnitude or direction of the original flux that matters. It is whether this flux is increasing or decreasing and the rate at which it changes.

> If the flux increases then $\mathbf{B}_D$ is in the opposite direction of $\mathbf{B}$.
> If the flux decreases then $\mathbf{B}_D$ is in the same direction of $\mathbf{B}$.

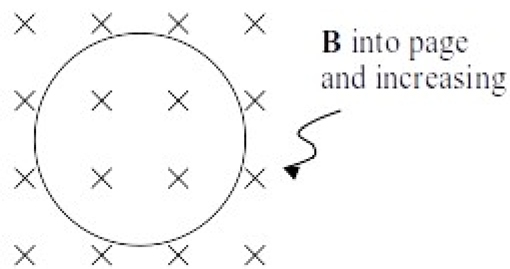
Once the direction of $\mathbf{B}_D$ is determined, the direction of I_D follows from the RHR of the Ampere Law.

The figure to the right illustrates how the flux in a loop can be changed by moving a bar magnet either toward or away from the loop. If there is no motion, as in (b), then there is no induced current. If you reverse the direction of the magnet, then the induced direction of the current is reversed.



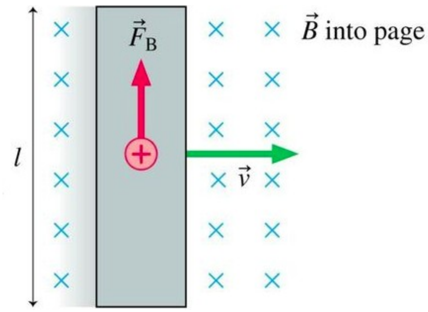
Example

The direction of the induced current must be so that it produces an induced magnetic field which is out of the page and opposes the increase. For this to occur, the direction of the induced current must be *counter-clockwise*.



The motional emf $\mathcal{E}_M$

For a single straight conducting rod moving perpendicular to a **B** field, the Lorenz force F_B moves the charges toward the upper end of the rod and leaves negative charge at the bottom.
The work done by Lorentz force is



$$W = \int F_B dy = F_B l$$

where F_B is assumed to be constant and l the length of the rod.

A potential difference is generated between the ends of the rod which is calculated as the work per unit charge.

$$\mathcal{E}_M = \frac{F_b l}{q} = \frac{qvBl}{q} = Blv$$

If we were to connect the opposite ends of the rod with a wire, then we would have a complete circuit and current will flow.

Example

A Boeing 747 has a wingspan of 60 m and is flying north at 250 m/s. The earth's magnetic field is 2.5 x 10^{-5} T and is directed north but below the horizon at 30º. What is the emf that is developed between the wing tips? What is the polarity of the emf?

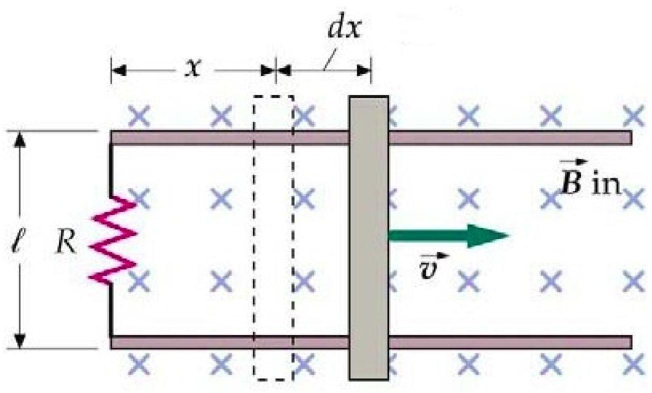
Only the vertical component of the magnetic field generates a motional emf. So,

$$\mathcal{E}_M = B\ l\ v = B\sin(20^0)\ l\ v = (2.5x10^{-5}T)\sin(30^0)(60m)(250m/s) = 0.19\ V$$

Using the right-hand rule, the force on the positive charges in the airplane are west, to the tip of the left wing is positive and the tip of the right wing is negative.

The 'Flux Rule'

Suppose the rod above is sliding along parallel rails that is connected at one end so that there is a complete circuit.



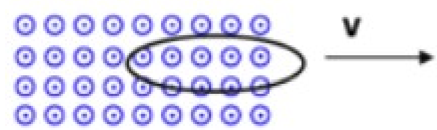
As the rod moves to the right with velocity *v*, the area enclosed by the circuit increases in time as $dA = l\ dx$. The magnetic flux through the loop increases. We can explain the motion of the charges in a differ way: using Flux rule which looks identical to Faraday Law:

$$\mathcal{E}_M = B\,l\,v \;=\; B\,l\,\frac{dx}{dt} \;= B\frac{dA}{dt} \;=\; \frac{d\Phi(B)}{dt}$$

Note how the flux changes because the area is time dependent. **B** stays constant.

Example

As a loop is moved out of the constant magnetic field **B** which points out of the page. The flux decrease because the area ‘facing’ **B** decreases. An induced current is generated in the loop because of the motional emf (the Lorentz force acting on the charges present in the wire).



What about the direction? Since the flux is decreasing the induce magnetic field points in the same direction of **B**, from Ampere Law the induced current is counter-clockwise.

When the flux rule is used the motional emf $\mathcal{E}_M$ (due to the Lorentz force acting on a wire moving in an external constant **B** field) and the induced emf $\mathcal{E}_D$ (Faraday’s law), coincidentally have the same form and describe the same effect: the induced current in the wire.

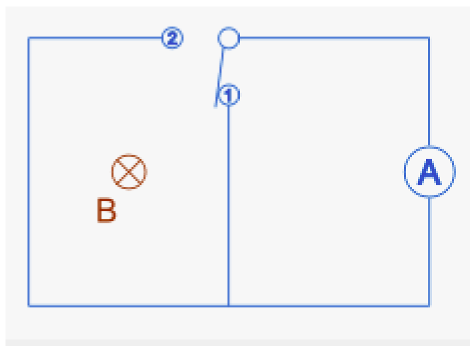
To physically disgusting the two keep in mind: if the magnetic flux changes because the $B(t)$ field is changing in time, that is the Faraday Law. If the flux changes because the area $A(t)$ or the angle $\theta(t)$ (see next) is changing in time, then it is the flux rule.

The flux rule can be used to calculate the motional emf $\mathcal{E}_M$ but with caution: it might not work if switches or sliding contacts or extended conductor (Eddy currents) are present. To get the correct answer for $\mathcal{E}_M$ it is always safer to start from the Lorentz force acting on the wire.

Example - Faraday’s Paradox

A circuit is placed in a region where a uniform B field points into the page. Faraday Law does not apply in this case since **B** does not change in time. What about the flux rule? When the switch is moved from position 1 to 2, the area of the circuit doubles. If we were to apply the flux rule we would obtained a non-zero emf_M

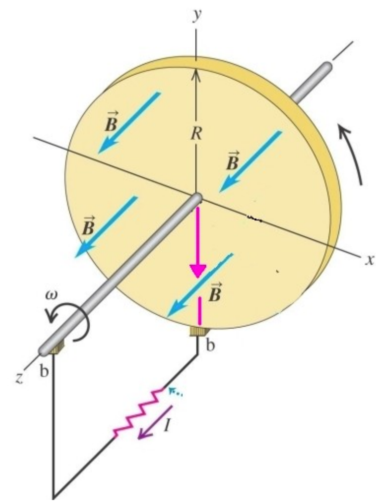
$$\mathcal{E}_M = \frac{d\Phi(B)}{dt} = B\frac{dA}{dt}$$



from which it follows an induce current in the circuit. In reality this is not what happens, the ammeter measures no current. Why? Because no force is applied to move the charges in the wire, and so there is no Lorentz force to generate a potential difference.

Example – Faraday Disk

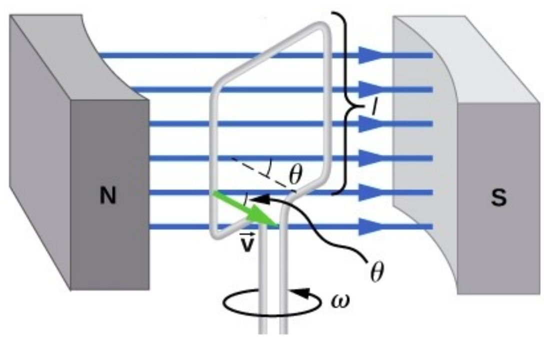
A conducting disk rotates with velocity ω in a region where a uniform B field is present. The Faraday Law does not apply in this case since **B** does not change in time. A sliding contact allows for a close circuit (charges move radially from the center of the disk to point *b*) and there exists an induced current. Since there is no change in the flux thought the loop made with the wire, the flux rule would wrongly give a zero induced current. The current originated from the motion of charges in a B field (Lorentz law).



The flux can changed in a coil by changing the area or the orientation of the loop in the field, i.e. if there is a change in the direction between **B** and **n** (the normal to *A*).

Example

A loop of areas *A* rotates with angular velocity ω in a uniform **B** field. The angle θ between **B** and **n** changes because of a force acts on the loop. The green arrow represents the velocity of a charge inside the wire. Because of the Lorentz force a motional emf drives the induced current in the loop.
Alternatively, using the flux rule (which can be used in this case)



$$\mathcal{E}_M = \frac{d\Phi}{dt} = BA\frac{d}{dt}\cos(\theta)$$

Example

Suppose the magnetic field of the previews example is kept at 0.5 T, but the loop is flipped by 180° in a time of 0.05 s. What would be the induced emf?

The flux would reverse direction, so the change would be twice the original value.

$$\Delta\Phi = BA\,\Delta(\cos\theta) = BA(\cos\pi - \cos 0) = -2\,BA$$

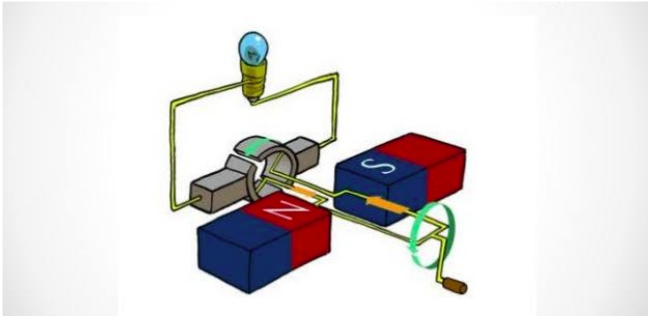
$$|emf_M| = \frac{\Delta\Phi}{\Delta t} = \frac{2(0.5\,T)\pi\,(0.02)^2}{0.05\,s} = 0.025\,V$$

Electric generator

An electric generator converts mechanical energy (motion) into electrical energy (voltage and current). It's explained by the flux rule. It works like a reversed DC motor.

Example

A coil with N turns and area A rotates at constant angular frequency f within an uniform **B** field. What is the expression of the induced emf?



Since the angular velocity $\omega = 2\pi f$ is constant, then $\theta(t) = \omega t$

$$\mathcal{E}_M = N\frac{d\Phi}{dt} = N\,B\,A\,\frac{d}{dt}\cos(\omega t) = \omega NBA\sin(\omega t)$$

The *emf* depends on the frequency. Its max value, corresponding to $\sin(\omega t) = 1$

$$\mathcal{E}_M = \omega NBA$$

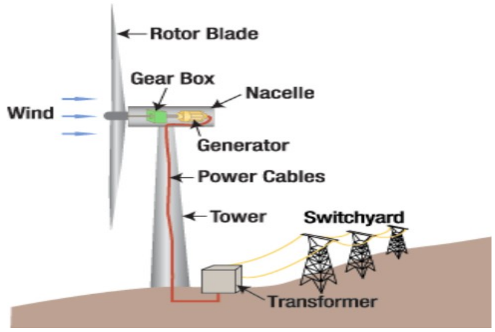
depends on the rotational frequency. If the rotation happens faster then the emf_{MAX} increases.

An electric generator works because of the Lorentz force which pushes the charges to move (the induce current). In this sense it can be seen as an electric motor working in reverse.

Electric Power plant

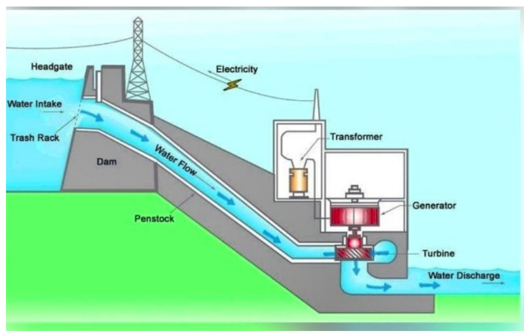
Wind:

The motion of the wind rotates the turbine.



Hydroelectric:

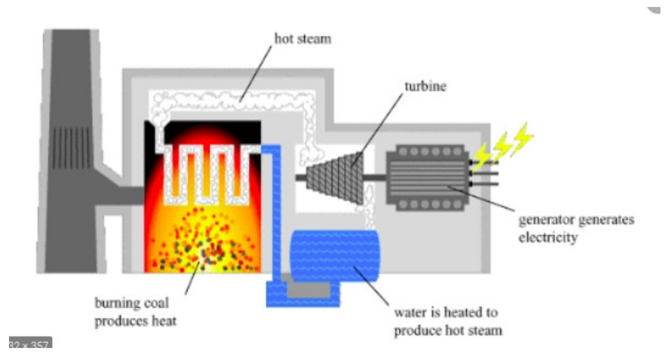
The water transfer its potential energy into mechanical energy to rotate the turbines.



Fossil Fuel:

The heat boils water and generates a stream of steam which rotates the turbine.

Very bad for the environment. Good only for the greedy people who do not care to destroy the planet for profits.



We do have the technology to go 100% green using renewable energy. Politics is the reason why is not the reality (yet).

The General Form of Faraday Law

A **B** field is present in a region of space. The **B** field changes in time (for example a time dependent current is the source of it). As a consequence an electric field is generated in the surrounding. Since $\Delta V = -\int \vec{E} \cdot d\vec{l}$ we obtain the relation between the **E** and **B** as

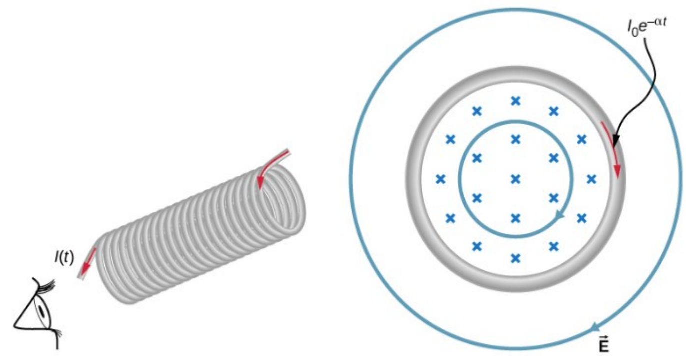
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\,\Phi(\vec{B})}{dt} \qquad \text{(integral form)}$$

The integration is a circulation (close path) which does not depend on the path. This form of Faraday Law is more general since it holds without requiring any wire. If then you place a wire, each charge experiences the force $F = qE$ and their motion generates the induced current I_D.

The direction of **E** is obtained by checking the direction of the induce current I_D if a wire is placed over the circulation.

Example

A current $I(t) = I_0 e^{-\alpha t}$ is the source of a **B** field inside a solenoid. Find the expression of the electric field at distance r outside the solenoid.



Because of Faraday Law an electric field is generated since **B** varies in time. The change is the flux is

$$\frac{d\Phi(B)}{dt} = \alpha\mu_0 n I_0 A e^{-\alpha t}$$

with $B(t) = \mu_0 n I(t)$ for a solenoid with cross section A. Along the circulation at a fixed r, the E field is constant and can be taken out of the integral

$$E\oint dl = E\ 2\pi r$$

from which it follows

$$E(r,t) = \frac{\alpha\mu_0 n I_0 A}{2\pi}\frac{1}{r}e^{-\alpha t}$$

We notice how the electric field is both time and position dependent.

There is an important difference between the induce electric field and the electric field due to charge distribution. The latter in conservative

$$\oint \vec{E}\cdot\vec{dl} = \int_a^b \vec{E}\cdot\vec{dl} + \int_b^a \vec{E}\cdot\vec{dl} = 0 \Rightarrow \int_a^b \vec{E}\cdot\vec{dl} = \int_a^b \vec{E}\cdot\vec{dl} \quad \text{(electrostatics)}$$

where the last two integration, having the same limits *a*,*b* are evaluated over two different paths. This means the integration is path independent and the field is conservative. This implies a potential can be defined as $dV = -\vec{E}\cdot d\vec{l}$. The electric field does not do work in moving a charge from *a* back to *a.*

In the case of varying B field we have instead

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d\,\Phi_B}{dt} \neq 0 \quad \text{(induced)}$$

which implies the electric field in not conservative, it does work over a closed loop and there is no potential such as $dV = -\vec{E}\cdot d\vec{l}$.

The Sources of the Electric and Magnetic fields

The general form of Faraday Law relates a geometrical propriety of the electric field with the change of the magnetic flux which generates it. If we considers only a point *P*, there is no circulation, no loop and no flux through it. More fundamentally Faraday Law tells us that an electric field is generated at *P* if a B field is varying in the surrounding.

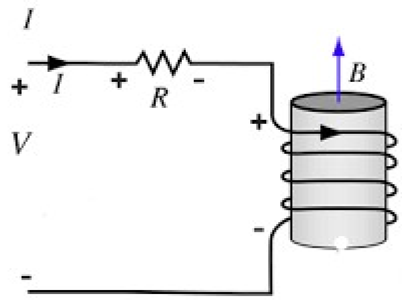
The following table summarize the possible sources of **E** and **B** we have learned so far

sources of **E**	*Q the charge*	$dB/dt \neq 0$
sources of **B**	*I the current*	

If you’re wondering why the empty cell, Maxwell gave the answer. It is in the following chapter on electromagnetic waves.

Inductance

Consider the circuit below, in which you have a battery connected in series with a resistor and a coil (e.g., a solenoid) of N turns. From Ampere Law



I = constant

If the flux in each loop is Φ_B then the total flux is $N\Phi_B$. This total flux is proportional to the current in the coil (Ampere Law).

$$N\Phi_B = LI$$

The constant of proportionality L is called the *self-inductance* or simply *inductance*

$$L = \frac{N\Phi_B}{I}$$ unit = Wb/A = henry (H)

If the coil is solenoid then we can derive its inductance in terms of its geometrical proprieties

$$N(BA) = LI$$
$$N\left(\mu_0 \frac{N}{l} I\right) A = LI$$

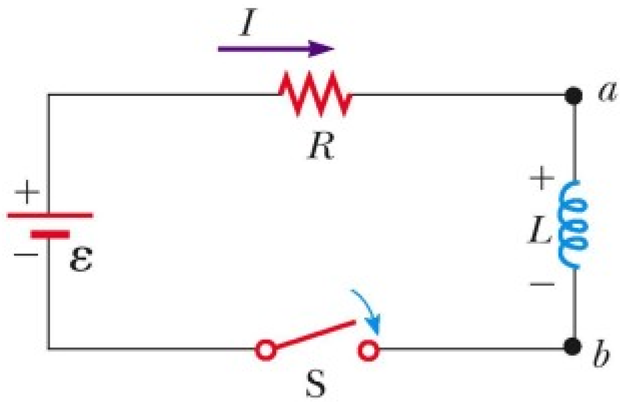
which imply $L = \dfrac{\mu_0 N^2 A}{l}$ (inductance of a solenoid)

Example

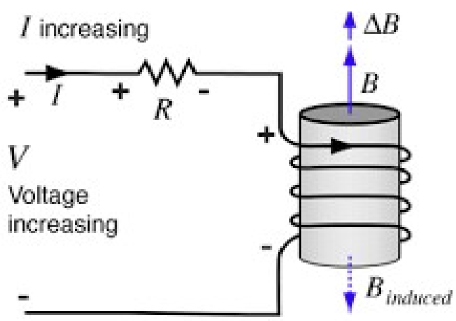
A solenoid has 100-turns of wire, a diameter of 2 cm, and length l = 5 cm. What is its inductance?

$$L = \frac{\mu_0 N^2 A}{l} = \frac{(4\pi\ x\ 10^{-7})(100)^2 \pi (0.01)^2}{0.05} = 7.9 x 10^{-5} H$$

Now, let's consider a non constant current I(t). For example add a switch in the *RL* circuit



When the switch (S) is closed, the current through the coil starts to increase from zero to some final steady-state value. This changing current produces a changing magnetic field and a changing magnetic flux in the coil. According to Faraday's law, an emf will be induced in the coil whose direction will oppose the change taking place. The polarity of the emf (a potential difference between the two ends of the coil) will be as shown in the figure. It will oppose the current that the battery is attempting to deliver to the circuit and slow its increase to its final steady-state value.



I = increasing

According to Fararday's law and the above definition of L, the emf of the inductor is given by

$$emf_L = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

There is a emf_L across the inductor only while the current is changing.

Solving the *RL* circuit

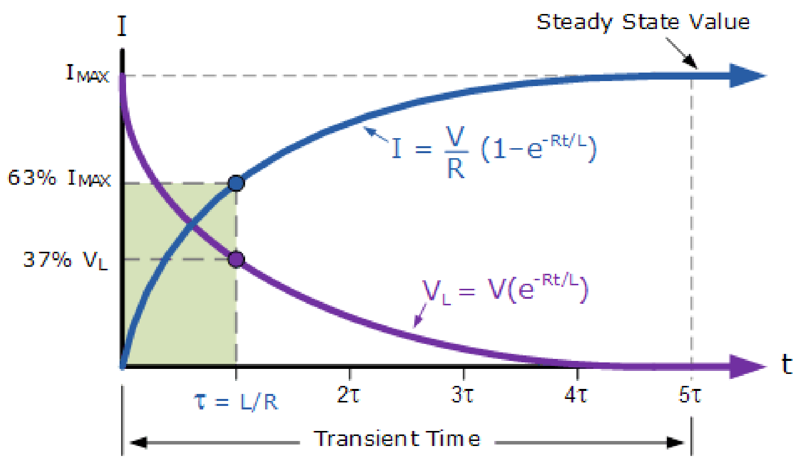
Applying Kirchhoff rule at the one loop of the *RL* circuit

$$V - IR - L\left(\frac{dI}{dt}\right) = 0$$

where V is the emf provided by the battery. Solving the differential equation for I(t)

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

The current increases exponentially in time, somewhat like the voltage across a capacitor as it is charged.



After it reaches a steady state, there will be no emf_L and the current is simply

$$I_{MAX} = \frac{V}{R}\ .$$

The time constant is

$$\tau = \frac{L}{R}$$

Note τ for the *RL* circuit varies inversely with R, whereas it is proportional to R for an *RC* circuit ($\tau = RC$).

The voltage across the inductor is

$$V_L = -L\frac{dI(t)}{dt} = V e^{-\frac{R}{L}t} \qquad \text{(see the figure above)}$$

Example

A *RL* circuit consists of solenoid L = 87.5 mH a resistor R = 250 Ω and a battery. Find the time for the current to reach half of its steady value after the switch is closed.

$$\frac{1}{2}I_{MAX} = I_{MAX}\left(1-e^{\frac{R}{L}t}\right) \ \Rightarrow \ \frac{1}{2} = \left(1-e^{-\frac{R}{L}t}\right)$$ and solving for t

$$t = \tau \ \ln 2 = 0.243\, s$$

<u>Energy in a inductor</u>

An inductor carrying a current contains energy in the form of the magnetic field. The energy can be calculated from the power delivered by the current, $P = \text{emf}_L\, I$. The work done in increasing the current by dI is

$$dW = Pdt = V_L I \ dt = L\frac{dI}{dt}Idt = LI\, dI$$

The total work done to increase the current to I, which is the magnetic potential energy stored in the inductor, is

$$PE_L = \int dW = \tfrac{1}{2}LI^2$$

This potential energy is the analog expression of the potential energy stored in a capacitor ($PE_C = ½\, C\Delta V^2$) due to the electric field between the plates.

Example

If the current in the inductor is of the previous example is 2 A, what is the stored energy?

$$PE = \tfrac{1}{2}LI^2 = \tfrac{1}{2}(7.9\,x\,10^{-5})(2)^2 = 1.58\,x\,10^{-4}\,J$$

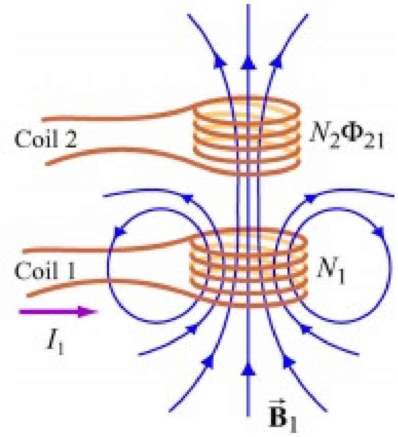
If this inductor is de-energized through a 50-Ω resistor, what is the time constant?

$$\tau = \frac{L}{R} = \frac{7.9\,x\,10^{-5}}{50} = 1.6\,x\,10^{-6}\,s = 1.6\ \mu s$$

Mutual Inductance

Two coils are place next to each others. Coil 1 has N_1 turns, coil 2 has N_2 turns.

Coil 1 carries a current I_1 which generates $\mathbf{B}_1$.
If I_1 is time dependent, an induce *emf* is present in coil 2



$$emf_2 = -N_2 \frac{d}{dt}\Phi_{21}(B_1)$$

since $\Phi(B_1)$ is proportional to I_1 we can write

$$emf_2 = -M_{21}\frac{dI_1}{dt}$$

The constant of proportionality M_{21} is called the *mutual inductance.*

By equating the two expressions of the emf_2 we obtained

$$N_2 d\Phi_{21}(B_1) = -M_{21} dI_1$$

and after integrating both sides

$$M_{21} = \frac{N_2\Phi_{21}}{I_1} \qquad \text{(unit henry)}$$

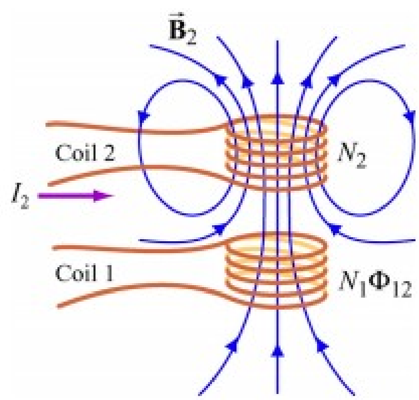
Note: M_{21} depend on the geometrical proprieties of the two coils.

If we repeat the argument but starting with a current I_2 the coil 2 we obtain

$$emf_1 = -M_{12}\frac{dI_2}{dt_1}$$

and

$$M_{12} = \frac{N_1\Phi_{12}}{I_2}$$

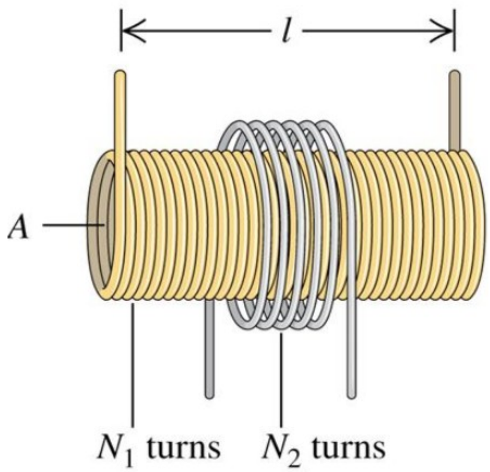


Using the *reciprocity theorem* (based on Ampere and Biot-Savart laws) it can be shown

$$M_{21} = M_{12} = M$$

Example

Two coils are coupled as shown in the figure, find the mutual inductance M



Let's consider the inner coil to be a solenoid of cross section A_1 and for which we have $B_1 = \mu_0 \frac{N_1}{l} I_1$

$$M_{21} = \frac{N_2 \Phi_{21}(B_1)}{I_I} = \mu_0 \frac{N_1 N_2 A_2}{l}$$

where A_2 is the cross section of the larger coil. Since $B_1 = 0$ outside the solenoid, the flux through A_2 is the same as the flux through A_1 and we can write

$$M_{21} = \mu_0 \frac{N_1 N_2 A_1}{l} \quad .$$

Note how the mutual inductance depends only on the geometrical proprieties of the system.

If we were to start by considering a variable current I_2 in the outer coil we have

$$M_{12} = \frac{N_1 \Phi_{12}(B_2)}{I_2}$$

where Φ_{12} is the flux of B_2 through the inner solenoid of area A_1. Now this flux can be quite complicated to calculate since the outer coil is not a solenoid and its B_2 field has a complicated geometry. Nevertheless because of the reciprocal theorem we know that $M_{12} = M_{21}$.