Chapter 13 - Universal Gravitation

In Chapter 5 we studied Newton's three laws of motion. In addition to these laws, Newton formulated the law of universal gravitation. This law states that two masses are attracted by a force given by

$$F = \frac{Gm_1 m_2}{r^2},$$

where G = 6.67 x 10^{-11} N·m^2/kg^2 (<u>not</u> g = 9.8 m/s^2). For spherical masses, r is the distance between the centers of the masses.

The weight of a mass, m, on the surface of earth is then

$$W = F = \frac{GM_E m}{R_E{}^2},$$

where R_E is the radius of the earth. Since we can also write W = mg, then

$$g = \frac{GM_E}{R_E{}^2}.$$

<u>Example</u>:

Calculate g from the above formula.

$$g = \frac{(6.67x10^{-11} N \cdot m^2 / kg^2)(5.98x20^{24}\, kg)}{(6.38x10^6\, m)^2} = 9.8\, m/s^2$$

At a distance of 6.38 x 10^6 m above the surface of the earth, g = 9.8/(2)^2 = 2.45 m/s^2 and a person's weight would be ¼ that on the surface.

Kepler's laws of planetary motion

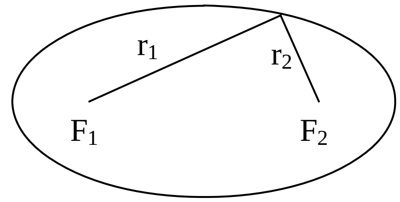
Well before Newton's time, Johannes Kepler formulated his three laws of planetary motion. Kepler deduced these 'laws' by carefully studying data on planetary motion obtained by Tycho Brahe. Isaac Newton was able to explain Kepler's laws from the laws of motion and the law of universal gravitation.

Kepler's laws are

1) Planets move in elliptical orbits with the sun at one of the focal points.
2) A line from the sun to a planet sweeps out equal areas in equal times.
3) The square of the orbital period of the planets is proportional to the cube of the average distance from the planet to the sun.

Kepler's 1st law:

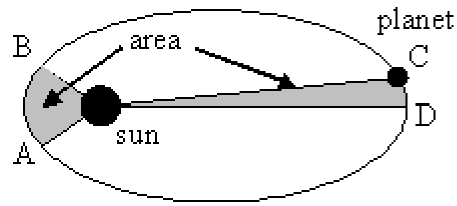
It can be shown mathematically from Newton's law of gravitation that an object will orbit a second massive object in an elliptical path unless the energy of the orbiting object is so great that the orbit is not closed. An ellipse is an oblong closed curve with two focal points, as shown to the right. The ellipse can be traced out by following the path such that the sum of the distances from the focal points to any point on the curve is constant. That is, $r_1 + r_2$ = constant. In the case of a planet orbiting the sun, the sun is at one of the focal points of the ellipse.



A circle is a special case of an ellipse where the two focal points are the same.

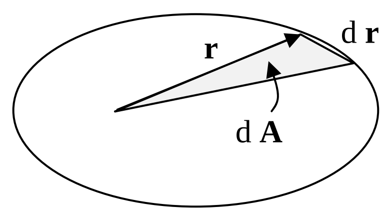
Kepler's 2nd law:

The figure to the right is meant to illustrate Kepler's 2nd law. A planet goes in an elliptical path around the sun. The time to move from A to B is the same as the time to move from C to D. From A to B the planet moves faster than from C to D so that the area swept out by the line from the planet to the sun is the same for both time intervals.



Kepler's 2nd law is a direct consequence of conservation of *angular* momentum and the fact that the force of attraction is directed alone the line connecting the two bodies.

The area swept out by the vector from the sun to the planet in a small time dt is



$$dA = \tfrac{1}{2} r\, dr_\perp = \tfrac{1}{2} r\, v_\perp dt = \tfrac{1}{2} r\, \omega\, r\, dt$$

where $r_\perp$ and $v_\perp$ are the components perpendicular to the vector **r** connecting the two bodies.

Thus, $\dfrac{dA}{dt} = \tfrac{1}{2} r^2 \omega$

The angular momentum of the planet about the sun is

$$L = I\omega = m r^2 \omega,$$

or, $r^2\, \omega = \dfrac{L}{m}$

Thus,

$$\frac{dA}{dt} = \frac{L}{2m}$$

The angular momentum of the planet, L, is constant since the sun does not exert a torque on the planet. This is because the force exerted by the sun is along the line connecting the sun and the planet. Thus, dA/dt = constant, which means that the planet sweeps out equal areas in equal times.

Kepler's 3rd law:

The 3rd law can most easily be obtained for a circular orbit as follows.

$$F = ma_c$$

$$\frac{GM_s m}{r^2} = m\frac{v^2}{r} = m\frac{(2\pi r/T)^2}{r} = m\frac{4\pi^2 r}{T^2}$$

where M_s = mass of sun, m = mass of planet, T = period of orbit of planet, and r = sun-planet distance. Solving the above for T^2, we have

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3$$

This equation would apply for any object orbiting a fixed body. For example, for a satellite orbiting the earth, M_s would be replaced by M_E, the mass of the earth.

(Note: In the case of an elliptical orbit, Kepler's 3rd law still applies as above except that the radius of the circle is replaced by the semi-major axis of the ellipse.)

Example:

What would be the period of a satellite in orbit just above the surface of earth? (Of course, such an orbit could not be sustained because of atmospheric resistance.)

$$T^2 = \left(\frac{4\pi^2}{GM_E}\right)R_E{}^3 = \left(\frac{4\pi^2}{(6.67x10^{-11})(5.98x10^{24})}\right)(6.38x10^6)^3$$

$$T = 5{,}070\,s = \underline{85\,\text{min}}$$

What is the speed of a satellite in a circular orbit?

$$F = m\frac{v^2}{r} = \frac{GM_E m}{r^2}$$

or,

$$v = \sqrt{\frac{GM_E}{r}}$$

For such a low earth orbit,

$$v = \sqrt{\frac{(6.67x10^{-11})(5.98x10^{24})}{6.38x10^6}} = 7{,}900\, m/s = 17{,}700\, mph$$

Questions: How does the period of orbit change as the radius increases? How does the satellite speed change with increasing radius? How does the period or orbit depend on the mass of the satellite?

Gravitational potential energy:

From the gravitational force formula, one can obtain a general expression for the potential energy, *U*, of two attracting masses. In general, the change in potential energy is the negative of the work done by the conservative force, so

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F dr = \int_{r_i}^{r_f} \frac{Gm_1 m_2}{r^2} dr = -\frac{Gm_1 m_2}{r_f} + \frac{Gm_1 m_2}{r_i}\ .$$

By convention, the potential energy in universal gravitation is taken to be zero when r = ∞. Thus,

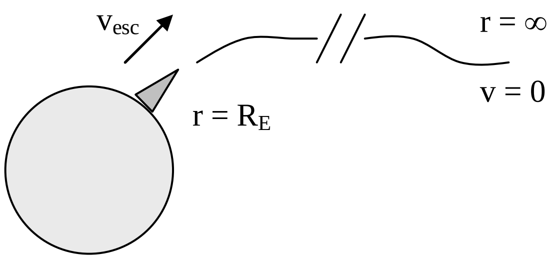
$$U(r) = -\frac{Gm_1 m_2}{r}$$

This expression is more general than the expression ΔU = mgΔy, which is valid for values of Δy that are small compared to earth's radius. For finite separations, U for universal gravity is always negative; however, in applications we only worry about changes in U, which can be positive or negative.

Example:

What is the 'escape' speed of an object from a planet?

By escape speed, we mean the minimum speed to launch an object such that it never returns to the surface of the planet. This would require that it go an infinite distance from the planet where it eventually comes to rest. We use conservation of energy.



$$E_i = E_f$$

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}mv_{esc}^2 - \frac{GM_E m}{R_E} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{r_f} = 0 - 0 = 0$$

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

For earth, the escape speed is

$$v_{esc} = \sqrt{\frac{2(6.67x10^{-11})(5.98x10^{24})}{6.38x10^6}} = \underline{1.12x10^4\ m/s} \quad (= 25{,}000\,mph)$$

(The actual ‘escape’ speed is larger because of atmospheric resistance.)