Lecture 9: Schwarzschild solution

Quick recap

To begin, let's recap what we learned from the previous lecture. There were a lot of abstract concepts and sophisticated mathematics displayed, so now would be a good time to summarize the main ideas. Again, the point is not to be able to understand the details with extreme rigor, but to grasp the concepts.

Inertial reference frames: In the absence of gravity, IRFs were represented by straight worldlines and the interval between two events was given by the flat metric equation, $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. In the presence of gravity we observed that IRFs no longer follow straight lines but are represented by curved lines. The Principle of Maximal Aging stated that reference frames in which Newton's first law was valid (IRFs), the proper time between two events is greatest for such a worldline compared to all other worldlines connecting the same two events. However, in the presence of gravity the curved worldline represents the IRF, and hence have the maximal proper time, whereas the flat metric gives that the straight worldline has the maximal proper time. The conclusion drawn was that the flat metric is no longer appropriate to describe events in a gravitational field.

Geodesics: These worldlines of IRFs are unique, in that there is only one worldline between two events which has the maximal proper time. (There are an infinite number of lines with shorter proper time). The worldlines are special, and we gave them the name geodesics. The analog with space is that geodesics in space are the unique lines giving the shortest distance between two points. On a curved surface (sphere) we observed that the geodesic between two points is not always a straight line (as represented on a flat map, it actually is the straightest line). If we can describe the geodesic mathematically for an object, then this is equivalent to solving Newton's second law to determine the trajectory of this object. The geodesic is the solution.

Curvature: In four dimensional space it is impossible to visualize a curved three or four dimensional surface. We examined curved two dimensional surfaces to get an idea of the techniques used to determine curvature. These techniques can then be used to check for curvature of three or four dimensional spaces. In two dimensions, we discussed the Gaussian intrinsic curvature. Forget the formula, the argument that a circles circumference will differ from $2\pi R$ is enough to get a handle on it. In fact, you can define the Gaussian curvature with such arguments (see previous days notes). The demonstration involving tossing the five colored balls within a gravitational field clearly showed that the geodesics become curved. Hence, we will need to discuss four dimensional curvature to describe gravity.

Metric: The metric equation is the main entity we have been studying these two weeks. The metric equation in flat spacetime is simply the definition of the squared interval above. Examining some simple lower dimensional curved surfaces, we saw that the metric equation differed for these spaces. Explicitly, we demonstrated that for the metric equation on a sphere, $dl^2 = R^2d\theta^2 + R^2\sin^2\theta d\phi^2$, it was the $\sin\theta$ term which did not allow us to return to the starting point of the square journey. Thus telling us that this is a curved space. To reiterate, some of the important metric equations we discussed were the following, (R is a constant value, r is a variable),

2D Euclidean Plane in polar coordinates (r, θ): $dl^2 = dr^2 + r^2d\theta^2$
2D surface of a sphere (spherical coordinates): $dl^2 = R^2d\theta^2 + R^2\sin^2\theta d\phi^2$
3D Euclidean Space in spherical coordinates: $dl^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$
4D Flat spacetime (Cartesian coordinates): $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
4D Flat spacetime (Spherical coordinates): $ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$

The metric specifies the entire local geometry of the surface. The metric also specifies the geodesics. For example, to find the geodesics of light rays (the paths they will follow) we set the interval between two events to be zero. The remaining equation can be written as a function of a line in spacetime describing the worldline of the light ray. Getting the geodesics out of the metric is related to finding the curvature. We can use the trajectories of geodesics to determine the curvature. (If two nearby parallel geodesics start to deviate and later are far apart, then there is curvature in the space).

The next step was to generalize this. If we consider a fairly complex space, then in two and three dimensions the metric can be expressed in the following way,

2D: $dl^2 = g_{11}dx_1^2 + g_{22}dx_2^2$
3D: $dl^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2,$

where the g_{ii}s can be functions of the coordinates and the x_is are generalized coordinates (in different situations we call them different things). (We are also not considering 'off-diagonal' terms such as $g_{12}dx_1dx_2$). If we look at the metric equations given above, we can express these in the following manner,

2D Euclidean Plane in polar coordinates (r, θ): $g_{rr} = 1, g_{\theta\theta} = r^2.$
2D surface of a sphere (spherical coordinates): $g_{\theta\theta} = R^2, g_{\phi\phi} = R^2\sin^2\theta.$
3D Euclidean Space in spherical coordinates: $g_{rr} = 1, g_{\theta\theta} = r^2, g_{\phi\phi} = r^2\sin^2\theta.$
4D Flat spacetime (Cartesian coordinates): $g_{tt} = 1, g_{xx} = -1, g_{yy} = -1, g_{zz} = -1$.
4D Flat spacetime (Spherical coordinates): $g_{tt} = 1, g_{rr} = -1, g_{\theta\theta} = -r^2, g_{\phi\phi} = -r^2\sin^2\theta.$

The set {g}are simply the coefficients of the squared difference of the variables.

Again, we see that the metric is the center of the whole game. Given the metric is equivalent to be given the geodesics, i.e. the solution. The remaining piece of the puzzle is to know what form the metric takes, i.e. the set of functions {g}.

Einsteins equation: This is the last piece of the puzzle. Einsteins equation gives the relation between the energy density and the curvature of spacetime.

The 'tensor' $G^{\mu\nu}$ is a sophisticated way to represent the complex nature of curvature of spacetime. (Imagine at single point there are four perpendicular directions. In each combination of directions (x-t, x-y, x-z, y-t, y-z, etc.) there can be a separate radius of curvature.) The tensor $T^{\mu\nu}$ is a sophisticated way to represent energy densities, momentum densities (pc), and combinations thereof. The Einstein tensor is a complex function of the metric functions {g}. Solving these equations is a rather complex business.

We will not be able to explicitly solve for any solutions in this course. We will introduce the first solution developed and try to convince you that it has the correct form.

Schwarzschild Solution

Within a month of the publication of Einsteins General Theory of Relativity, Karl Schwarzschild found a solution for a very simple system. (Schwarzschild died within a year due to illnesses from World War I). This solution does describe a good number of the commonly occurring situations that one is interested in. The solution is for the case of a spherically symmetric mass, M, and considering all points outside of this mass. (Known as the exterior solution, the interior solution is much more complex). This case approximates much of the scenarios that occur within our solar system for example, and can be used to solve the problem with the advance of the perihelion of Mercury.

Some properties which this solution will have and also some constraints we should impose on it:

1. Since it is spherically symmetric, we will use spherical coordinates to describe the spatial part of the metric.
2. Since this solution is outside of the mass, Einsteins equation takes the form, $G^{\mu\nu} = 0$. (Note just because there is no mass outside does not mean that there is no curvature).
3. Condition 1: As we take the distance from this mass out to infinity ($r \to \infty$) the metric should approach the flat spacetime metric, $ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$.
4. Condition 2: As the mass is taken to zero, we should again regain the flat spacetime metric.
5. Condition 3: We have spherical symmetry. This will impose limits upon the form of the metric.
6. Lastly, it must satisfy Einstein's equation.

Well, we could apply some equivalence principle arguments to find the basic form of one of the components of the metric, however we do not have time to present the derivation. So the most general form of the metric is of the following form,

$$ds^2 = g_{tt}c^2dt^2 - g_{rr}dr^2 - h_\theta r^2d\theta^2 - h_\phi r^2\sin^2\theta d\phi^2.$$

where $h_\theta = g_{\theta\theta}/r^2$ and $h_\phi = g_{\phi\phi}/(r^2\sin^2\theta)$. Since we have spherical symmetry we can immediately set $h_\theta = h_\phi = 1$. If these differed from 1, then angles would be different for different values of r or θ. We must have the same metric for these last two terms as we had for the metric for a 2D sphere. This leaves us with,

$$ds^2 = g_{tt}c^2dt^2 - g_{rr}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Also because of spherical symmetry we must have $g_{tt} = g_{tt}(r,t), g_{rr} = g_{rr}(r,t)$. All that is left is to find these two functions which requires solving Einsteins equation. What we will do is to state the result and see if it satisfies our conditions above. The solution is,

Hence, our metric equation outside of a static, spherically symmetric mass M is,

$$ds^2 = (1 - \frac{2GM}{rc^2})c^2dt^2 - \frac{1}{(1 - \frac{2GM}{rc^2})}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the Schwarzschild metric.

This equation gives us the geometry of spacetime outside of a single massive object. We could use the Earth, Sun, or a black hole by inserting the appropriate mass. As this metric is the correct one to use in situations within the Earth's gravitational field, the effects discussed before, gravitational time dilation and gravitational red shift, can be derived from this metric equation. If we use the Sun as the mass, then the bending of starlight around the Sun and the advance of the perihelion of Mercury can be found from this metric equation. Extending this to a black hole (forthcoming) we can describe any bizarre effects near black holes.

Does it satisfy our conditions? Examine the form when $r \to \infty$. It should be clear that both g_{tt} and $g_{rr} \to 1$. We regain the flat spacetime metric. Examine the form when $M \to 0$, again g_{tt} and $g_{rr} \to 1$. We regain our flat spacetime metric when we 'turn off' the mass.

Another interesting thing to note. Neither g_{tt} nor g_{rr} depend upon time. What this says is that the solution is static for a spherically symmetric body[1]. Even if the mass is pulsating (remaining at mass M but its radius may be changing), the exterior metric remains static. This tells us that spherically symmetric bodies can not produce gravitational waves. It takes a spherically asymmetric situation to produce such waves.

Understanding the Schwarzschild metric

Now that we have the form of the metric, what do we get from it. Firstly, we can get information about geodesics. This is a little advanced for us, so we will stick to some more simple observations.

Let's first examine what the three dimensional space looks like with this new metric. Again, it is difficult to view a three dimensional curved space so we will restrict our view to a plane. In addition, since the metric is static, we can just view it at one particular time and this will be the same at all other times.

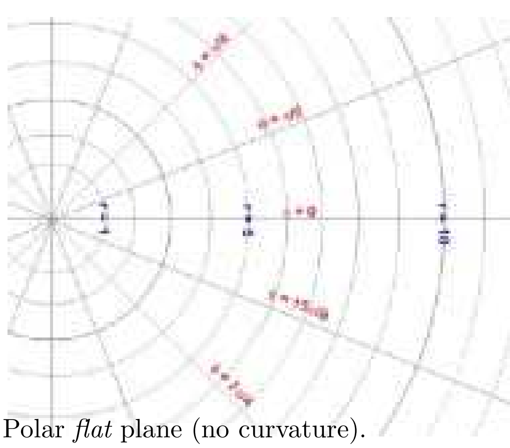
So we take a time slice, (set t = constant so that $dt = 0$), and set $\theta = \frac{\pi}{2}$. (Recall from spherical coordinates that this is just examining the equatorial slice). We have gone from a four dimensional spacetime down to a two dimensional space. The metric with these restrictions is now,

$$ds^2 = -\frac{dr^2}{(1 - \frac{2GM}{rc^2})} - r^2 d\phi^2.$$

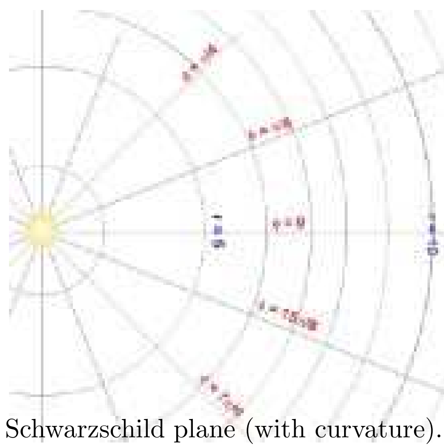
In order to relate this to measures we've seen before, lets flip the signs and call this dl^2,

$$dl^2 = \frac{dr^2}{(1 - \frac{2GM}{rc^2})} + r^2 d\phi^2.$$

Stare at this formula for two minutes. What are the features? First of all, the second term is nothing more than what we had for a 2D polar plot. Again, this just verifies the spherical symmetry. However in the radial direction there is a change. Let's plot this out.



Polar *flat* plane (no curvature).

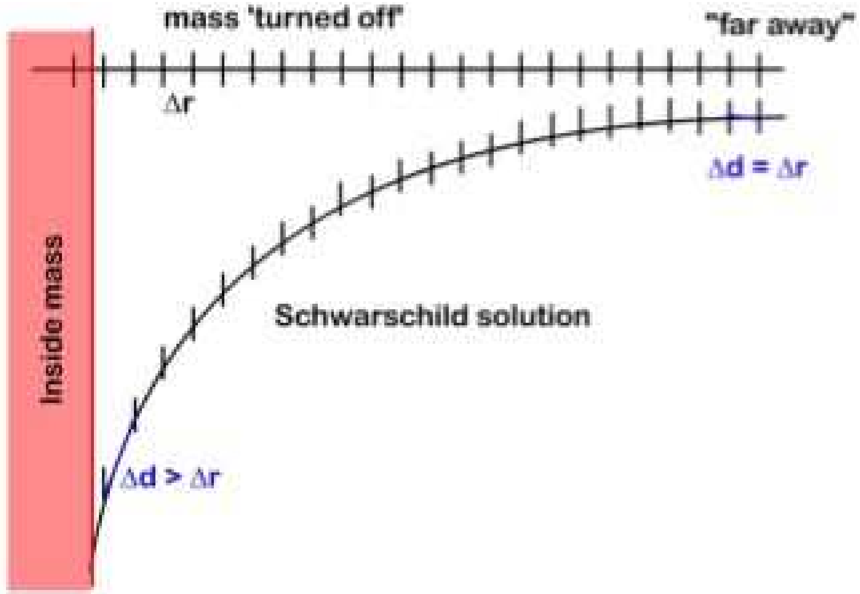


Schwarzschild plane (with curvature).

If we go out a long way, this measure will just be the same for a two dimensional plane in polar coordinates. As we draw closer to the mass, the term in parentheses gets smaller (r gets smaller, $\frac{2GM}{rc^2}$ gets larger which is being subtracted off of 1). Now taking the inverse, this coefficient is getting larger. What does this mean? If we attempt to plot this on a flat piece of paper we have 1m = 1m far away, and as we draw closer the radial distance increases. (Place an observer far away, she measures 3 ly to the object (somehow, since she does not travel this distance). Now have an object actually lay down rulers to get to the mass, the measurement will be larger, say 5 ly). We can not draw these increasing radial distances as we approach this object on a plane. How can we picture them? Examine a side plot.

[1] This us called *Birkhoff's theorem.*

If the flat line represents the distance when the mass is 'turned off' (or measured within the far away observers coordinates*) then we see that Δd, distance measured off as the object is approached should be represented on the curved line. In this way we can fit our distances into this line.



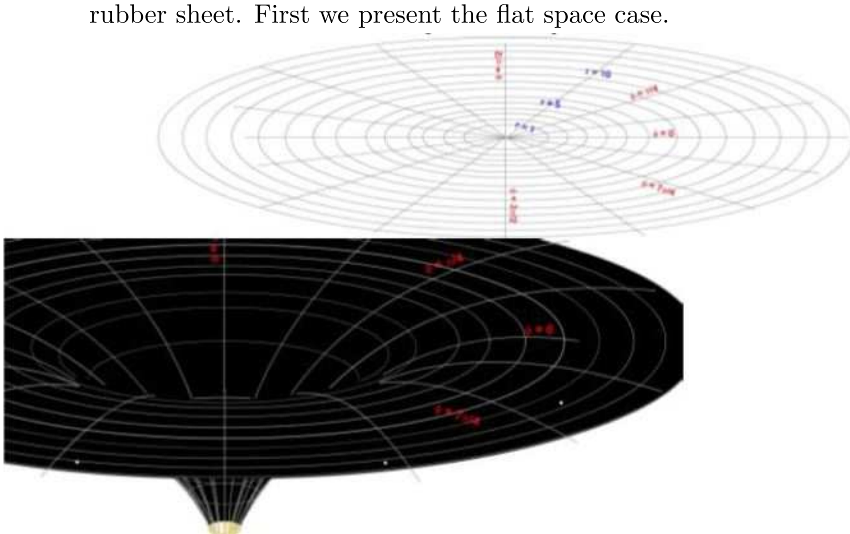
*Setting up a reference frame

Be very careful with the statement above (*). Since this observer remains far away, there is no such way that she can actually measure this distance out. Recall that to set up a reference frame to perform measurements we place friends with rulers and clocks at all of the intersections of the lattice points of space. However here, if we place friends at each 1 meter closer (and angularly) they measure off the distance dl which will vary as they approach.

There is another way to set up these shells such that there radii will agree with the coordinate r when the mass is 'turned off' (i.e. flat spacetime). How is this done? At one such position from the object, measure the circumference going around the object. Maintain the same distance from the object and travel around it, arriving back to your starting point. Then with the circumference measured, C, calculate the radius which would produce that circumference, i.e. $r = \frac{C}{2\pi}$. This distance r is the same radius you would measure if the space were not curved (mass 'turned off'). We can use this new measure of distance from the object, **the reduced radius** (or coordinate radius) or simply r, to label this shell. Repeat this process for regularly spaced reduced radii and you have a coordinate system involving r which agrees with the flat coordinate system (what the far away observer measures far away). We will call these observers, the shell observers. Hence, when we discuss how the space varies with r, we are referencing it in relation to the reduced radius. Again, the reduced radius simply being the distance one would measure if the mass were not there.

If one were to travel from one shell to the other, laying out a ruler, they would disagree that each shell is 1 meter apart. Of course they are not in the curved spacetime. This construction will allow us to relate the curvature of space to what we would observe in the absence of any mass.

This leads us to a very familiar plot, which I am sure you have seen several times. This is the idea of the stretched rubber sheet. First we present the flat space case.



Now if we place a mass we can picture it in the following manner (lower figure).

We want to explicitly state what this represents, as there are often misconceptions with such images. All we are doing is trying to fit our new coordinates into a visualization scheme. This represents a two dimensional space and that is it. It is not bending into a third dimension! (even though it appears to). This is intrinsic (not extrinsic) curvature.

Often you will see demonstrations involving rolling balls on a stretched rubber sheet. Be careful in what this is depicting. First, the ball can not lie on top of the sheet, the universe is represented by the two dimensional sheet and that is all! No third dimension. We are creatures lying within this sheet. Second, as the ball is rolling around it is doing so under the influence of gravity, again in three dimensions. There is no higher dimensional form of gravity making things move the way they do, simply the natural trajectories of objects in spacetime. In this sheet, draw the straightest line between two points, this is the geodesic. (Of course, to be proper we should include the time dimension).

To extend this picture to three dimensions, picture changing the above image to all possible angles. From no matter which direction you approach the mass the distance gets longer as you approach (compared to the shell observers). Of course, you cant completely picture this in your mind, but you can get an idea.

Understanding the Temporal Nature of the Schwarzschild Metric

Now that we have pictured the static lower dimensional view of the spatial (radial) portion, lets examine what the coefficient of the time coordinate is telling us. Let's compare a clock at infinity (say Bob's clock) to a clock located near this object at a shell coordinate r away (Alice's clock). Consider two events at the position of Alice. Her clock ticks off two ticks, this we can call her proper time between ticks, $\Delta\tau$. Since these two events occur at the same place, $dr = d\phi = 0$. We then have,

$$\Delta\tau \quad = \quad \Delta t_{shell} = \sqrt{1 - \frac{2GM}{rc^2}}\Delta t \qquad\qquad \longrightarrow \quad \Delta t_{shell} < \Delta t$$

Compare this with our previous discussion of gravitational time dilation. There we said, via the Equivalence Principle, that Bob (who was higher up in the gravitational field) observed Alices clock to run slower. This agrees with that result. Alice observes her clock to tick every second ($\Delta\tau$), Bob observes Δt seconds between each tick, $\Delta\tau < \Delta t$. We now have a quantitative measure of gravitational time dilation (as well as length expansion).

Black Holes, introduction

The idea of an object so massive that light can not escape is not a new idea at all. In 1783 John Michell conceived of such an object and in 1798 Laplace expanded on the idea. The argument follows along classical lines. Lets do a classical calculation. Recall that the escape velocity for an object trying to leave an object of mass M and radius R is determined by setting the gravitational potential energy equal to the kinetic energy (in this way for the object to just reach infinity with velocity zero requires an amount of energy equal to the gravitational potential energy).

$$\frac{GMm}{r} = \frac{1}{2}mv^2 \quad \longrightarrow \quad r = \left.\frac{2GM}{v^2}\right|_{v=c} = \frac{2GM}{c^2} \equiv R_S$$

We see that for an object of mass M and radius R_S, the escape velocity is equal to the speed of light. For any radius less than this the escape velocity is greater. Hence no object, nor information can escape from such an object. However, there is a difference between this classical idea and what GR predicts. Classically if the escape velocity is equal to the speed of light implies that a light ray can not reach out to infinity. This is what the definition of the escape velocity means. It can go most of the way out, but then falls back into the object. So this classical model is only partially correct, it gives the correct radius but that is about it.

The Schwarzschild metric can be used for the exterior of any spherically symmetric object. With it we can find geodesics for very light test particles, (particles which do not significantly curve spacetime). However there is a problem, observe what happens to the coefficients when $r \to \frac{2GM}{c^2}$. The coefficient of the temporal term vanishes and the radial coefficient becomes infinite. We see there is a problem here. Since this is still an exterior solution, this demands that the object in question must have a radius $r < \frac{2GM}{c^2}$. This corresponds to a very dense object. Since these coordinates breakdown at this distance, we give it a special name, the Schwarzschild radius, $R_S = \frac{2GM}{c^2}$. What is the significance of this radius? First, it is the breakdown point for the coordinates, or metric, we have used. The matter within such an object, once compressed down to this size, is no longer stable and continues to collapse down to a point, the **singularity**. Again our Schwarzschild metric tells us nothing of what goes on inside of this radius. We need to construct a new metric within. Second, the trajectory of a light ray trying to escape can not even go beyond this radius if emitted from within. Contrast this with Laplaces hole, there light came out and fell back in, here the light can not escape at all!

Typical sizes of Schwarzschhild radii:

	Radius now	R_S
Milky Way Galaxy	$50,000\ ly$	$0.005\ ly$
Sun	$7 \times 10^8\ m$	$3000\ m$
Earth	$6.4 \times 10^6\ m$	$8\ mm$

100 kg person	1 m	$1.5 \times 10^{-25}\ m$
H atom	$\sim 1. \times 10^{-10}\ m$	$2.5 \times 10^{-54}\ m$

Because of these sizes we do not observe everyday Earth bound objects forming black holes. The densities reached are enormous. [Problem]. Generally very massive stars (> 5 solar masses), and collections of stars (at centers of galaxies) form black holes. The formation of a black hole begins when a massive start runs out of fuel to carry on fusion reactions (there are actually several phases through which this occurs and some metastable states in between) and eventually collapses to a radius less than the Schwarzschild radius. After this point is a black hole (the term was coined in 1967 by John Wheeler).

Since the escape velocity is greater than c, no thing or information can escape from a black hole. (Hawking radiation will be discussed in the next lecture, a possible way for black holes to radiate away a small portion of its mass). However, there are some basic facts about the star which remain detectable outside of it. There are certain physical conservations laws that are always held true. Among them are conservation of electric charge (electric charge is never created nor destroyed), conservation of energy, and conservation of momentum and angular momentum. If a star has a net electric charge, after it collapses into a black hole, this net electric charge must still be detectable to the outside, similarly for its angular momentum. The value the spinning star had prior to collapse will still be detectable after its collapse. (Note: spinning black holes are not described by the Schwarzschild metric, another metric the Kerr metric (1963) is used. In addition, electrically charged, spinning black hole requires the Kerr- Newman metric (1965)). Of course the mass of the black hole is detectable by the amount of curvature of spacetime around it, (it appears in the metric).

This fact that only the mass, electric charge, and angular momentum of a star survives after its collapse into a black hole is given by the phrase "Black holes have no hair". Unfortunately, we will not be able to explore these more exotic type of black holes due to time. They are more mathematically sophisticated and have interesting properties, but we will stick to the, strange enough, nonspinning spherically symmetric black hole.

Observations for Objects Entering Black Holes

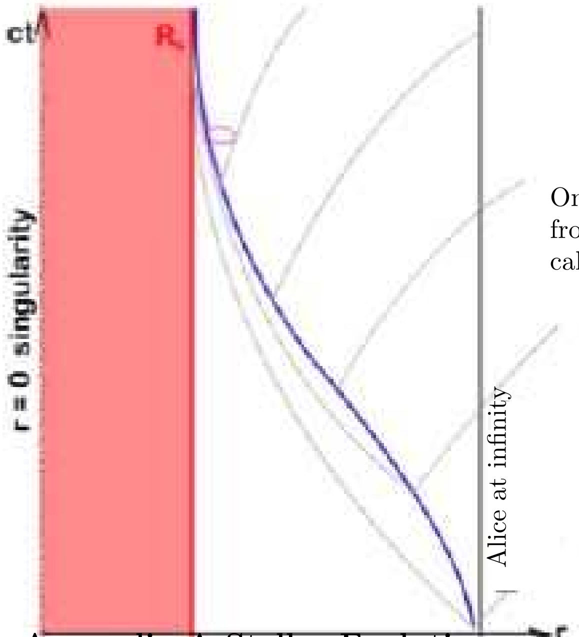
First we want to describe what Alice observes (or the network of shell observers) as Bob plunges into a black hole. So we drop Bob off from rest at a very large distance away from the black hole. Initially both Alice and Bob agree as to their rulers and clock rates. As Bob falls in, say one half of the way, Alice's shell observer friends notice that his clock is running slower as compared to Alice's. As the journey takes several years, Bob does not age as quickly as Alice. As he approaches ever closer to the horizon (Schwarzschild radius) his clock continues to run slower. In addition, the image of Bob falling in as Alice sees it gets continually redshifted, eventually she will need a radio telescope to observe him. From the metric we see that as $r \to R_S, \Delta t \to 0$. Alice never sees Bob pass through the horizon, she basically observes him clock to stop, he is frozen at the horizon forever. Of course, she would have difficulty seeing him as the light reaching Alice becomes infinitely redshifted. Again any light emitted after Bob passes the horizon never reaches outside of the horizon. Once inside, $r' < R_S$, he must travel to the singularity.

Of course there are several other things that would most likely occur before Bob crossed the horizon. First, tidal forces would rip him apart lengthwise. The force on his feet will eventually pull so much more than his head that he would eventually become a stream of atoms, then a stream of electrons and protons, etc. In addition, usually around black holes (most of which are spinning) there is a large amount of matter falling in. As this matter heats up and spins at greater and greater rates, it releases vast amounts of radiation (gamma rays – which are highly lethal). So poor Bob would be suffering from radiation poisoning as he is being ripped to shreds. General advice, avoid a black hole if you can. Note that there is nothing terribly dangerous when you are a distance away from a black hole. The gravitational force of the black hole is the same as the star was prior to collapse for distances $r >> R_S$. For example, if the Sun were (somehow) to instantaneously turn into a black hole, the orbit of the planets would be unaffected. Here on Earth, nothing would change except it would be night time all day.

Crossing the horizon

We have examined what happens as Alice views Bob as he plunges into the black hole. What happens from Bob's viewpoint? Especially as he crosses the horizon. Assuming he can (somehow) avoid the problems discussed in the previous paragraphs, Bob will reach and cross the horizon without fanfare. In his reference frame all is pretty much normal (outside of the enormous tidal forces trying to rip him apart). His clock continues to tick at one second intervals and he continues to measure one meter meter sticks (in his free falling frame). (*Note that the size of the free falling frame gets very small as he approaches so he will eventually come to the conclusion that he is not in an IRF). For him the Schwarzschild event horizon is no big whoop, he just passes on through, nothing special happens. (Of course he can't get out, but that actually occurs earlier). Once inside, we can not compare what happens to the outside because we can not create a network of shells inside since all objects, once within, proceed to the origin.

What actually occurs inside the horizon can be discerned by changing to a new coordinate system more suitable to Bobs journey. Doing so, it is seen that the light cones tip forward such that all timelike geodesics intersect the origin. (I.e. there is no stable orbits within the horizon – all goes to the singularity). (This will be discussed in more detail in the next lecture).



One last plot will show the effect of Bob approaching the horizon from the outside. The geodesic in this spacetime plot asymptotically approaches and becomes parallel to the Schwarzschild radius.

Appendix A Stellar Evolution

So what does it actually take to create a black hole? The full answer is rather involved, here we will provide a simple description of how stars evolve. It turns out that not all stars will evolve to become black holes. Our Sun will not evolve into a black hole but a different type of object.

Stars are hot and bright due to nuclear fusion processes within them. In the most basic process, hydrogen nuclei fuse together to form helium and releases a large amount of energy in doing so. (The actual nuclei involved is a little more complex). This process is the same as that which occurs in a thermonuclear weapon. In addition to releasing a large amount of energy and light a shock wave prevents the stellar material from collapsing upon itself. Very massive stars would quickly collapse into a black hole if it were not for the powerful fusion processes occurring within.

Stars only contain a finite amount of hydrogen to burn in this process. Eventually mostly helium will be left. Fortunately, helium can under fusion to heavier elements (listed as primarily carbon in the diagram). This process continues to progress to heavier and heavier elements until iron (Fe) is reached. Iron is the most stable of nuclei and will not fuse to form heavier nuclei. (Heavier nuclei split apart under nuclear fission). Once a star reaches the point where most of it is comprised of iron, no more fusion processes can occur and the star begins to collapse upon itself under its own gravitational attraction. At this point one of three end points is reached depending upon the original mass of the star. In proceeding through this sequence up to this point, stars will undergo many transformations. Our Sun for example will (many million years in the future) expand in size possibly up to the orbit of Mercury. Many stars will undergo violent explosions, or novae (and the much more powerful supernova), as they implode. After all of these steps there are three possibilities,

A: White Dwarfs

B: Neutron Stars

C: Black Holes

A: Stars collapse down until all of the atoms wavefunctions begin to overlap. There is an important result of quantum mechanics, the Fermi Exclusion Principle, which forbids (spin $\frac{1}{2}$) objects to exist in the same state in the same place. This provides a mechanism of resistance to the implosion pressures. This mechanism is called electron degeneracy pressure. Many stars will end their evolution at this point, a white dwarf. The white dwarf is very hot (due to the compression) and dense. One teaspoon of the material comprising a white dwarf would weigh (on Earth) as much as an elephant. It was calculated by Subrahmanyan Chandrashekhar that a white dwarf will be stable provided its mass is less than 1.4 solar masses. Stars originally of 8 solar masses and less form white dwarfs (the remaining mass is blown off in nova events). If the object has a mass greater than the Chandrashekhar limit then its gravity will overpower the electron degeneracy and continue its collapse to form a neutron star.

B: The electron degeneracy pressure now overwhelmed, the next resistance is provided by the nucleons within the nuclei of the matter. It is a similar reason (nucleons and electrons both being spin $\frac{1}{2}$ particles) for why the electrons provided resistance. Now the density is far greater. Under the immense pressure the atoms (electrons +protons) combine into neutrons (in a process called reverse beta decay). All that is left are neutrons providing resistance due to the Fermi Exclusion Principle. However, if the star has a very large mass, even this resistance is overcome and the object becomes a black hole.

C: There are now no more processes left to provide resistance. Gravity wins out over all else and the object continues to collapse until it becomes a singularity (at this point our theories breakdown and a theory of quantum gravity is required to describe the very center of a black hole). It is not easy to give a strict limit but basically stars which are generally more than 10 solar masses become black holes.