Chapter 3
The 4-Dimensional World View

Now he has departed from this strange world a little ahead of me. That means nothing. People like us, who believe in physics, know that the distinction between past, present, and future is only a stubbornly persistent illusion.

—Albert Einstein

3.1 Introduction

We have seen the basic physical consequences of the two postulates of Special Relativity and we know how to derive them from the mathematical transformation relating two inertial frames, the Lorentz transformation. The mathematics gives us an insight into how space and time are inextricably mixed and the most natural way to see this is in a representation of the world with four dimensions, three spatial and one temporal. This is not just mathematics: the physics just makes so much more sense when viewed in four dimensions than in three spatial dimensions with time as a parameter (because that's all time is in Newtonian physics). In this chapter we begin to study the geometry of the 4-dimensional continuum. The description of physics in this continuum requires the formalism of tensors, which is presented in the next chapter. After learning this formalism, we will study the kinematics and dynamics of point particles and systems of particles. However, it is not until Chap. 9 (when continuous distributions of matter and fields are introduced into the spacetime arena), that the power and elegance of the 4-dimensional world view are fully revealed.

V. Faraoni, *Special Relativity*, Undergraduate Lecture Notes in Physics,
DOI: 10.1007/978-3-319-01107-3_3,

3.2 The 4-Dimensional World

Let us compare the Galilei and the Lorentz transformations:

Galilei transformation	Lorentz transformation
$x' = x - vt,$	$x' = \gamma\,(x - vt)\,,$
$y' = y,$	$y' = y,$
$z' = z,$	$z' = z,$
$t' = t,$	$ct' = \gamma\left(ct - \frac{v}{c}\,x\right).$
Galilei transformation	**Lorentz transformation**
leaves Newtonian mechanics invariant	leaves Maxwell's theory invariant
intervals of absolute time are invariant	time intervals are not invariant
3-D lengths are invariant	3-D lengths are not invariant

Since the Lorentz transformation mixes the time and space coordinates, it implicitly suggests to treat these quantities on the same footing and to contemplate a 4-dimensional space (ct, x, y, z). The 4-dimensional world view was developed by Hermann Minkowski after the publication of Einstein's theory. In Minkowski's words,[1] "Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality".

In the Newtonian picture of the world, space and time are separate entities with time playing the role of a parameter in the Newtonian equations of motion. An *event* is now simply a point in the 4-dimensional Minkowski spacetime. If we consider two simultaneous events (t, x_1, y_1, z_1) and (t, x_2, y_2, z_2), the 3-dimensional *Euclidean distance* (squared)

$$l_{(3)}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \geq 0 \tag{3.1}$$

between the two spatial points is invariant under Galilei transformations. In the 4-dimensional view of the universe of Special Relativity, time and space merge into a continuum called *spacetime*. Given any two events (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2), the quantity that is invariant under Lorentz transformations is not the 3-dimensional length $l_{(3)}$ nor the time separation Δt between these events, but rather the 4-dimensional *spacetime interval* (squared)

$$\triangle s^2 \equiv -c^2\,(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\,, \tag{3.2}$$

[1] Reported from a famous 1908 lecture given by Minkowski at the Polytechnic of Zurich [1].

which is *not positive-definite* (in spite of the use of the symbol $\triangle s^2$). For infinitesimally close events $(ct, \mathbf{x})$ and $(ct + c\mathrm{d}t, \mathbf{x} + \mathrm{d}\mathbf{x})$, the *infinitesimal interval* or *line element* in Cartesian coordinates is

$$\mathrm{d}s^2 \equiv -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{3.3}$$

3-dimensional lengths and time intervals are *relative* to the inertial observer, while the 4-dimensional $\mathrm{d}s^2$ is *absolute* in Special Relativity; it is the same for all inertial observers related by a Lorentz transformation,

$$-c^2(\mathrm{d}t')^2 + (\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{3.4}$$

or

$$\mathrm{d}s'^2 = \mathrm{d}s^2, \tag{3.5}$$

as we already checked in Sect. 2.4. A 4-dimensional spacetime continuum equipped with the line element (3.3) which is invariant is called *Minkowski spacetime*. The background geometry for Special Relativity is the space $\mathbb{R}^4$ but not with the usual Euclidean notion of distance between points. The notion of distance needs to be generalized by the line element $\mathrm{d}s^2$ given by Eq. (3.3) because Lorentz transformations leave invariant $\mathrm{d}s^2$ but not the quantities $c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, $\mathrm{d}l_{(3)}^2$, or $\mathrm{d}t^2$. It is $\mathrm{d}s^2$ which is fundamental—in fact, the geometry of Minkowski space is even more fundamental than the theory of electromagnetism which led us to Special Relativity. Electromagnetic waves in vacuo, or zero-mass particles, and any signal which propagates at light speed satisfy the equation $c^2\mathrm{d}t^2 = \mathrm{d}\mathbf{x}^2$ or

$$\mathrm{d}s^2 = 0 \qquad \text{(massless particles)}. \tag{3.6}$$

Example 3.1 We are now ready to revisit our example of Chap. 1 in which a flash of light is emitted at the origin of an inertial system at $t = 0$ and, at time $t > 0$, is found on the surface of a sphere of radius ct. We have seen that the Galilei transformation to another inertial frame does not preserve this sphere, nor the equation $-c^2t^2 + r^2 = 0$. However, Lorentz transformations leave this equation invariant because it corresponds to $\Delta s^2 = 0$. Since the Maxwell equations are left invariant by Lorentz transformations, so is the propagation of light in this example.

* * *

We have already seen that the speed of light is a boundary that can never be crossed, or reached, by adding velocities smaller than c. Geometrically, this means that signals emitted at a spacetime point do not go out of the *light cone* emanating from that point, which is a surface generated by light rays passing through that point. In a two-dimensional (x, t) diagram using units such that $c = 1$, the light cone through a point is the locus of spacetime points forming two straight lines inclined at 45° and passing through that point (Fig. 3.1).

Massive particles are *subluminal*, i.e., they travel at speeds $v < c$; *luminal* particles travel at the speed of light c and comprise photons, gravitons, and possibly unknown particles with zero rest mass. *Superluminal* particles, also called *tachyons*, travelling at speeds $v > c$ have never been observed. Either they don't exist, or they don't interact with ordinary matter; they could travel backward in time. The trajectory of an hypothetical particle beconing tachyonic and exiting the light cone is depicted in Fig. 3.1. We will regard tachyon trajectories as physically forbidden. The fact that an event can only influence events lying inside or along its future light cone gives rise to the notion of *causality*. If particles which can become tachyonic and travel outside their light cone could exist, the notion of cause and effect would be in jeopardy.

Let O be a spacetime point; the set of points *causally connected* to O (i.e., the set of spacetime points which can affect O or be affected by O via signals propagating at speed c form the *light cone* or *null cone* through O. This surface comprises

- the *future light cone*, which is the set of all events that can be reached by O with light signals; and
- the *past light cone*, which is the set of all events that can influence O from the past by sending light signals.

We also identify

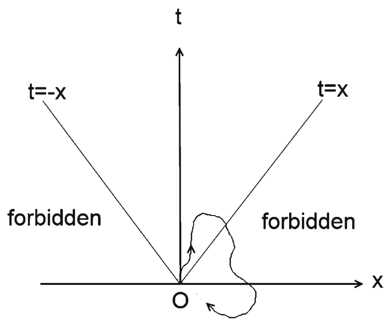

Fig. 3.1 The future light cone of the spacetime point O and a tachyonic trajectory. The region $|x| > |t|$ marked as "forbidden" cannot be reached from O using particles traveling at speeds $|v| \leq c$

- the *causal future* $J^+(O)$ of O, which is the set of all events that can be influenced by O through signals travelling[2] at speeds $v \le c$; and the
- the *causal past* $J^-(O)$ of O, the set of all events which can influence O from the past by sending signals travelling[3] at speeds $v \le c$.

Let S be a set of points O in Minkowski spacetime; then $J^\pm(S) \equiv \bigcup\limits_{O \in S} J^\pm(O)$ is the union of the causal futures [pasts] of the events O as O varies in S.

Although events that are simultaneous in an inertial frame are not simultaneous in other inertial frames, the notion of event A preceding event B cannot be altered by Lorentz boosts with $v < c$: if the time separation between two events is $\triangle t > 0$, it will be $\triangle t' > 0$ in any other inertial frame S'. The concept of causality is then Lorentz-invariant. All inertial observers agree that a certain event is in the absolute future or past of O because the time interval $\triangle t'$ between O and this event has the same sign of $\triangle t$ if $|v| < c$.

Formally, one can also consider the *elsewhere*, the complement of the causal past and causal future $J(O)$ of O. This set consists of all the events which cannot be connected with O by signals travelling at speed $v \le c$. Different observers disagree that an event in the elsewhere of O is in the past or the future of O. Events in the elsewhere of O have no causal connection with O.

3.3 Spacetime Diagrams

Consider an (x, t) diagram obtained by suppressing the y and z directions in Minkowski spacetime in Cartesian coordinates and setting $c = 1$ (*spacetime* or *Minkowski diagram*).[4] The space x at a constant time is represented by a straight line parallel to the x-axis (a "moment of time"), see Fig. 3.2. A point of space is represented by a vertical line of constant x (with the convention that one can only move forward in time, or upward along this line).

Consider another inertial frame S', which has axes x' and t': since $t' = \gamma\,(t - vx)$, the "moment of time" $t' = D$ in S' corresponds to

$$\gamma t - \gamma v x = D \quad \text{or} \quad t = vx + \frac{D}{\gamma}$$

in S, which is represented by a straight line with slope v and intercept at the origin D/γ. The x'-axis of equation $t' = 0$ corresponds to

[2] In the language of Chap. 5, this is the set of all events which can be connected with O by curves starting from O and have as tangent a future-directed causal vector.

[3] This is the set of all events which can be connected with O by curves ending at O which have as tangent a future-directed causal vector.

[4] See Ref. [2] for a detailed pedagogical introduction to spacetime diagrams.

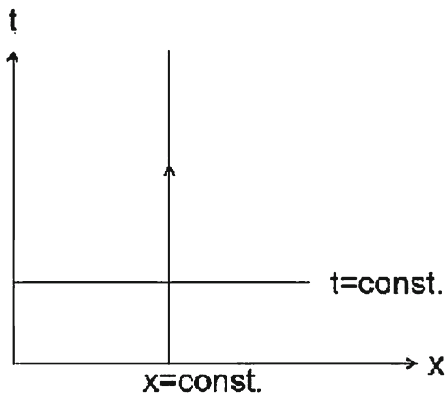

Fig. 3.2 The worldline of a particle which is stationary at x = const. in the inertial frame $\{t, x\}$, and a moment of time t = const

$$t = vx \qquad (t'\text{-axis}). \tag{3.7}$$

The line $x' = A$ in the S' frame corresponds to (using $x' = \gamma\,(x - vt)$)

$$\gamma x - \gamma vt = A, \qquad t = \frac{x}{v} - \frac{A}{\gamma v}.$$

In particular, the t'-axis of equation $x' = 0$ corresponds to

$$t = \frac{x}{v} \qquad (x'\text{-axis}). \tag{3.8}$$

The t'-axis and the x'-axis given by Eqs. (3.7) and (3.8), represented by lines with slopes v and $1/v$ which are the inverse of each other, are symmetric with respect to the diagonal $t = x$ (Fig. 3.3). The x'- and t'-axes can make any angle between 0° and 180°.

Since in these spacetime diagrams we use units in which $c = 1$, a light ray travelling at speed c is represented by a line at 45° with the coordinate axes. For example, two photons travelling along the positive or negative x-axis, respectively,

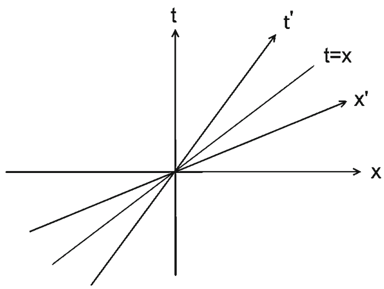

Fig. 3.3 Two inertial frames $\{t, x\}$ and $\{t', x'\}$. The x'- and t'-axes are symmetric with respect to the line $t = x$

and going through $x = 0$ at the time $t = 0$ are represented by the lines $t = \pm x$ (since objects can only travel forward in time, in the following we will often omit the arrows denoting the time direction).

A point of coordinates (x, t) in a spacetime diagram is an event; the history of a point-like particle is described by its spacetime trajectory, called a *worldline*. The history of an extended object is described by a *worldtube*, the collection of all the worldlines of the constituent particles. Worldlines in the (x, t) plane can be given by an equation of the form $t(x)$, or $f(t, x) = 0$, or by a parametric representation $\big(x(\lambda), t(\lambda)\big)$ in terms of a parameter λ (for massive particles, this is usually taken to be the proper time τ along the worldline). Since c is an absolute barrier in Special Relativity and no particle or physical signal travels faster than light, their worldlines have tangents with slopes larger than, or equal to, unity in the (x, t) plane.

Spacetime diagrams are useful to visualize simple processes in Special Relativity, for example, the emission of two consecutive light pulses from an inertial observer A and its reception and reflection back to A from a mirror located at B are described by Fig. 3.4, in which B is moving with constant velocity away from A in standard configuration.

The worldline of a massive particle which is accelerated in coordinates (x, t) will be a curve which is not straight (Fig. 3.5). Since $|v| < c$ at all times for a massive particle, the tangent to its worldline will always have slope larger than unity.

Since the squared interval is invariant under Lorentz transformations,

$$-c^2t'^2 + x'^2 = -c^2t^2 + x^2, \tag{3.9}$$

it is of interest to draw the hyperbolae (in units $c = 1$)

$$-t^2 + x^2 = \pm 1; \tag{3.10}$$

these curves coincide with the hyperbolae $-t'^2 + x'^2 = \pm 1$ and always intersect the axes at unit values of t, x, t', or x' (Fig. 3.6).

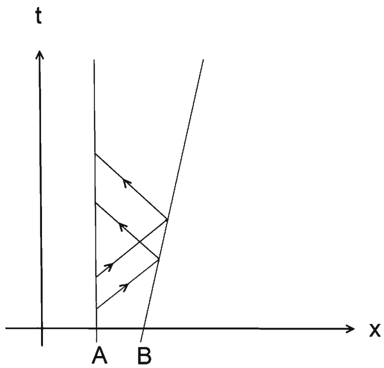

Fig. 3.4 Spacetime diagram of two light signals emitted by A, arriving at B, and reflected back to A. A is at rest in the (x, t) frame, while B moves with constant velocity

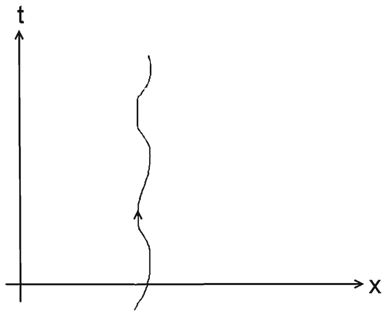

Fig. 3.5 The worldline of an accelerated particle

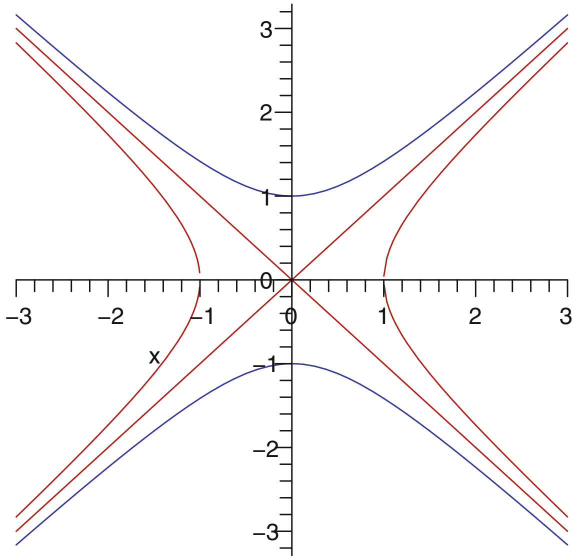

Fig. 3.6 The invariant hyperbolae $-t^2 + x^2 = -t'^2 + x'^2 = \pm 1$ asymptotic to the lines $t = \pm x$

The Lorentz transformation (2.1)–(2.4) allows us to conclude that:

- the x'-axis of equation $ct' = 0$ is the straight line $ct = \frac{v}{c} x$ of slope < 1 in units in which $c = 1$.
- The t'-axis of equation $x' = 0$ is the straight line of equation $ct = \frac{c}{v} x$ with slope larger than unity (when $c = 1$). This is the line symmetric to the x'-axis with respect to the diagonal $t = x$ in the (x, t) plane. It is also the worldline of an observer at rest in the $\{t', x'\}$ frame, which has zero 3-dimensional velocity in this frame but for which time keeps going on.
- The lines parallel to the t'-axis are the worldlines of particles at rest with fixed x' coordinate in S'. The lines parallel to the x'-axis are lines of constant t' (*lines of simultaneity* of S').

- The x'- and t'-axes make equal angles, but measured in opposite directions, with the line $t = x/c$ (in units $c = 1$, of course). As v gets closer and closer to c, these axes get closer and closer to the symmetry line $t = x/c$ in the spacetime diagram, and they merge with it in the limit $v \to c$. Formally this line is an invariant of the Lorentz transformation: $x' = ct' \quad \Leftrightarrow \quad x = ct$.

Note that the x'- and t'-axes do not appear perpendicular in the (x, t) plane in which the observer S' is not at rest. Also, the length scales along the axes are not the same. To relate a length on the x'-axis to that on the x-axis we use the invariant hyperbolae (3.10) (see Fig. 3.7). Since these hyperbolae intersect the axes at unit length, the intersection of this curve with the x-axis mapped into its intersection with the x'-axis gives the unit of length in S', which can be used to calibrate lengths. The same procedure applies to the time axes t and t'.

The coordinates $\left(x'_0, t'_0\right)$ in S' of an event of coordinates (x_0, t_0) in S can be obtained by projecting the event parallel to the coordinate axes x' and t' (Fig. 3.8).

It is now easy to understand graphically why two events that are simultaneous in the inertial frame S are not simultaneous in another inertial frame S' (Fig. 3.9). Simultaneous events according to S all lie on a horizontal line $t =$ const. in the (x, t) plane. Projecting two events P_1 and P_2 lying on this line parallel to the x'-axis yields two distinct intersections with the t'-axis, i.e., two events which are not simultaneous according to S'.

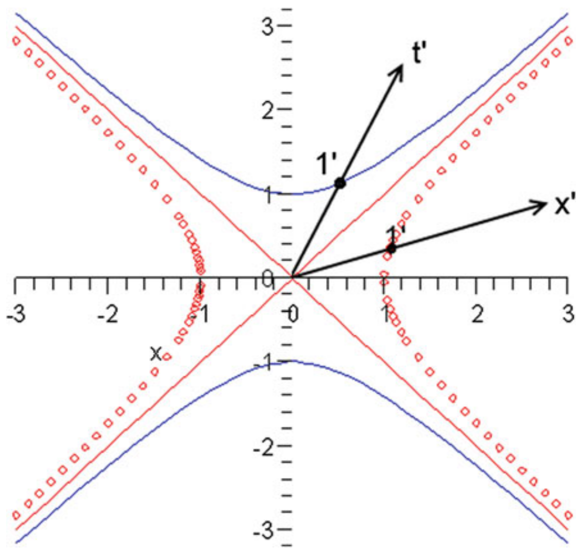

Fig. 3.7 A segment of unit length on the x-axis corresponds to a segment of different length (1') on the x'-axis, given by its intersection with the invariant hyperbola

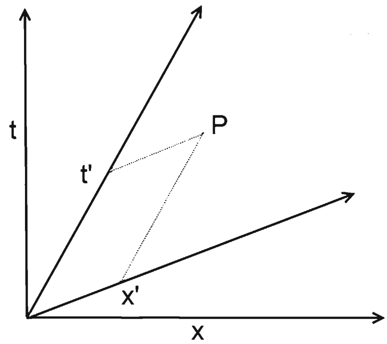

Fig. 3.8 The coordinates (x', t') of an event P in the inertial frame S' are obtained geometrically by projecting P parallel to the x'- and t'-axes

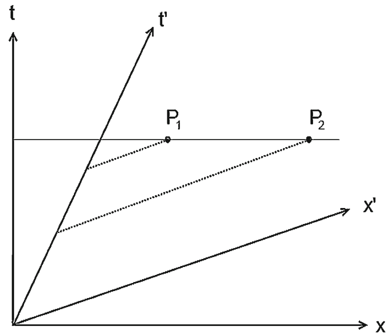

Fig. 3.9 The relativity of simultaneity in a spacetime diagram

3.4 Conclusion

The 4-dimensional world view provides much insight into the essence of Special Relativity, the intimate relations between space and time, the momentum and energy of a particle or of a physical system, the wave vector and the frequency of a wave, and the electric and magnetic fields. Before we uncover these relations, which we have just begun to see, we need to become acquainted with the necessary mathematics: the formalism of tensors.

Problems

3.1. Verify Eq. (3.5).

3.2. A particle moving with constant velocity with respect to a certain inertial observer O decays into two particles. Draw a spacetime diagram of the process in the reference frame of O.

3.3. Draw the causal future $J^+(S)$ and the causal past $J^-(S)$ of the set of events

$$S = \left\{ (x, t) : \qquad t = 0, \qquad x \in [0, 1] \cup [2, 3] \right\}$$

in 2-dimensional Minkowski spacetime. Write expressions for the boundaries $\partial J^+(S)$ of $J^+(S)$, $\partial J^-(S)$ of $J^-(S)$, and $\partial J(S)$ of $J^-(S) \cup J^+(S)$.

3.4. Draw the wordline of a particle moving with speed $c/2$ along the negative x-axis, and a series of light cones emanating from this particle at proper times $\tau = 1, 3$, and 6 seconds.

3.5. Consider two particles at rest along the x-axis at $x = x_1$ and $x = x_2$. Draw the past light cones of the events $(ct, x_1, 0, 0)$ and $(ct, x_2, 0, 0)$ with $t > 0$ in an (x, t) spacetime diagram. Under what conditions can the particles have interacted[5] at times $t \geq 0$?

3.6. Newtonian mechanics corresponds to the limit of Special Relativity when the speed of light becomes infinite (this is a formal limit, of course: c is in fact a constant). Discuss the causal structure of Minkowski spacetime in this limit: for example, what does the light cone through the origin of the (x, t) Minkowski spacetime become in this limit?

3.7. A particle oscillates along the x-axis with simple harmonic motion

$$x(t) = x_0 \cos(\omega t), \qquad y = z = 0$$

(where x_0 and ω are positive constants). Draw the particle worldline in an (x, t) spacetime diagram. Given the amplitude x_0 of the motion, what is the upper bound on its frequency?

3.8. Show that the line element

$$\mathrm{d}s^2 = -\frac{a^2 x^2}{c^4}\, c^2 \mathrm{d}t^2 + \mathrm{d}x^2,$$

where a is a constant with the dimensions of an acceleration and $x > 0$ and $t \in (-\infty, +\infty)$, is nothing but the line element of the 2-dimensional Minkowski spacetime in accelerated coordinates $\{cT, X\}$ given by

$$cT = x \sinh\left(\frac{at}{c}\right),$$

$$X = x \cosh\left(\frac{at}{c}\right).$$

Which portion of the 2-dimensional Minkowski spacetime is covered by the coordinates $\{cT, X\}$?

[5] This exercise is related to the *horizon problem* of Big Bang cosmology [3, 4].

References

1. H. Minkowski, translated by W. Perrett and G.B, Jeffery in *The Principle of Relativity* (Dover, New York, 1952)
2. T. Takeuchi, *An Illustrated Guide to Relativity* (Cambridge University Press, Cambridge, 2010)
3. C.B. Collins, S.W. Hawking, Astrophys. J. **180**, 317 (1973)
4. A. Liddle, *An Introduction to Cosmology* (Wiley, Chichester, 2003)

Chapter 5
Tensors in Minkowski Spacetime

Make everything as simple as possible, but no simpler.
—Albert Einstein

5.1 Introduction

After studying the general formalism of tensors, we can now apply it specifically to the 4-dimensional spacetime arena. A fundamental addition to the general baggage of tensors is the causal character of 4-vectors in Minkowski spacetime, which is due to the Lorentzian signature of the Minkowski metric. As a consequence of this signature, the line element is not positive-definite and we already know that this feature is linked to the existence of light cones, which play a crucial role because they determine the causal structure of the theory. Moreover, some specific notation which was not used in the general discussion of tensors applies to the Minkowski spacetime of Special Relativity.

5.2 Vectors and Tensors in Minkowski Spacetime

Let us consider now Minkowski spacetime and label the coordinates with the indices 0, 1, 2, and 3. The index 0 refers to the time coordinate. For example, in Cartesian coordinates, it is

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{5.1}$$

We adopt the convention that Greek indices assume the values 0, 1, 2, 3 and label spacetime quantities while Latin indices assume the values 1, 2, 3 and label spatial quantities. A contravariant vector in Minkowski spacetime is simply called a *4-vector*

V. Faraoni, *Special Relativity*, Undergraduate Lecture Notes in Physics,
DOI: 10.1007/978-3-319-01107-3_5,

and has components

$$A^{\mu} = (A^0, \mathbf{A}).$$

A^0 is the *time component* while A^1, A^2, A^3 are the *space components* which, together, form a 3-vector $\mathbf{A} = \left(A^1, A^2, A^3\right)$ with respect to purely spatial coordinate transformations $x^i \longrightarrow x^i(x^j)$ (here $i, j = 1, 2, 3$). However, $\mathbf{A}$ does not transform as a vector under 4-dimensional coordinate transformations.

Example 5.1 The position 4-vector is $x^{\mu} = (ct, \mathbf{x})$, and the gradient of a scalar function f is $\partial_{\mu} f = \left(\frac{\partial f}{c\partial t}, \nabla f\right)$.

A contravariant 4-tensor of rank 2 is a set of $4^2 = 16$ quantities that transform like the product of components of two 4-vectors

$$T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \frac{\partial x^{\nu'}}{\partial x^{\beta}} T^{\alpha\beta},$$

etc. The coordinate changes that we are interested in are mostly the Lorentz transformations between inertial frames with constant relative velocity (in standard configuration or otherwise), but one could transform from an inertial to an accelerated or rotating frame as well.

5.3 The Minkowski Metric

Let $\{x^{\mu}\} = \{ct, x, y, z\}$ be Cartesian coordinates in Minkowski spacetime. The spacetime interval between nearby points of coordinates $x^{\mu} = (ct, x, y, z)$ and $x^{\mu} + \mathrm{d}x^{\mu} = (ct + c\mathrm{d}t, x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{5.2}$$

We stick to this quantity because we know that it is left invariant by Lorentz transformations, which we have adopted as the fundamental symmetries of Special Relativity following the lesson coming from Maxwell's electromagnetism. By contrast, the usual Euclidean distance (squared) between two spatial points $\mathrm{d}l^2_{(3)} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, or their time separation (squared) $c^2\mathrm{d}t^2$, are not invariant under Lorentz transformations.

The infinitesimal interval, or *line element* of Minkowski spacetime $\mathrm{d}s^2$ can be obtained by introducing the metric tensor which, in Cartesian coordinates, has the components. [1]

[1] The symbol $\doteq$ denotes equality in a particular coordinate system.

$$
\left(\eta_{\mu\nu}\right) \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}\,(-1, 1, 1, 1) \tag{5.3}
$$

(*Minkowski metric in Cartesian coordinates*) and contracting it twice with the coordinate differentials $\mathrm{d}x^\mu$:

$$
\begin{aligned}
\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \begin{pmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} \\
&= -\left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 \\
&= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2,
\end{aligned}
$$

thus

$$
\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \doteq -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{5.4}
$$

In this sense, the Minkowski metric introduces the notion of distance between two points in spacetime. If the two points $x^\mu_{(1,2)}$ are not at infinitesimal distance, the finite spacetime interval in Cartesian coordinates is given by

$$
\triangle s^2 = -c^2\triangle t^2 + \triangle x^2 + \triangle y^2 + \triangle z^2, \tag{5.5}
$$

where $\triangle x^\mu \equiv x^\mu_{(2)} - x^\mu_{(1)}$.

The line element is not positive-definite because of the negative sign in the time-time component η_{00} of the Minkowski metric (5.3). This feature is absolutely necessary in order to introduce the notion of causality in Special Relativity and the notion of a limiting speed c. As already remarked, two points separated by a null interval $\mathrm{d}s^2 = 0$ can be related by a signal traveling at the speed of light:

$$
\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 = 0.
$$

Considering, for simplicity, a configuration such that $\mathrm{d}y = \mathrm{d}z = 0$ (points on the x-axis), it is $c^2\mathrm{d}t^2 = \mathrm{d}\mathbf{x}^2$ for this signal, or

$$\frac{dx}{dt} = \pm c$$

i.e., a signal travelling along the positive or negative x-axis and connecting two points $(ct, x, 0, 0)$ and $(ct + c\mathrm{d}t, x + \mathrm{d}x, 0, 0)$ with $\mathrm{d}s^2 = 0$ must necessarily travel at speed c, and vice-versa.

The Lorentzian signature $- + + +$ of the metric makes it clear that time is treated differently from space, although they both concur to build spacetime.
A metric tensor is positive-definite (or a *Riemannian metric*) if $g_{\mu\nu}x^{\mu}x^{\nu} > 0$ for any non-zero vector x^{μ}. Note that the convention on the metric signature is not unique. Several textbooks use the opposite signature $+ - - -$ for relativity, the physics being, of course, unchanged.

Example 5.2 Thus far, we have reasoned in Cartesian coordinates. However, it is clear that the components of a metric tensor will change if we change coordinates. Consider, as an example, the 3-dimensional space $\mathbb{R}^3$ with the Euclidean metric e_{ij} and the change from Cartesian to cylindrical coordinates $\{x, y, z\} \longrightarrow \{r, \varphi, z\}$

$$\begin{aligned} x &= r\ \cos\varphi, \\ y &= r\ \sin\varphi, \\ z &= z. \end{aligned}$$

We want to compute the components of the Euclidean metric tensor e_{ij} in these cylindrical coordinates. We begin by writing the Euclidean line element in $\mathbb{R}^3$

$$\mathrm{d}l^2_{(3)} = e_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

and by differentiating the inverse coordinate transformation $\{r, \varphi, z\} \longrightarrow \{x, y, z\}$,

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}r\ \cos\varphi - r\ \sin\varphi\mathrm{d}\varphi, \\ \mathrm{d}y &= \mathrm{d}r\ \sin\varphi + r\ \cos\varphi\mathrm{d}\varphi, \\ \mathrm{d}z &= \mathrm{d}z. \end{aligned}$$

Substituting into the expression of $\mathrm{d}l^2_{(3)}$, we obtain

$$\begin{aligned} \mathrm{d}l^2_{(3)} &= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = (\mathrm{d}r\cos\varphi - r\sin\varphi\mathrm{d}\varphi)^2 \\ &\quad + (\mathrm{d}r\sin\varphi + r\cos\varphi\mathrm{d}\varphi)^2 + \mathrm{d}z^2 \\ &= \mathrm{d}r^2\left(\cos^2\varphi + \sin^2\varphi\right) + r^2\left(\sin^2\varphi + \cos^2\varphi\right)\mathrm{d}\varphi^2 \\ &\quad -2r\ \sin\varphi\cos\varphi\,\mathrm{d}r\,\mathrm{d}\varphi + 2r\ \cos\varphi\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi + dz^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2. \end{aligned}$$

Since $\mathrm{d}l^2_{(3)} = e_{ij}\,\mathrm{d}x^i \mathrm{d}x^j$ is a tensor equation and is true in any coordinate system, we have that

$$\mathrm{d}l^2_{(3)} = \underbrace{e_{ij}\,\mathrm{d}x^i \mathrm{d}x^j}_{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2} = \underbrace{e_{i'j'}\,\mathrm{d}x^{i'} \mathrm{d}x^{j'}}_{\mathrm{d}r^2+r^2\mathrm{d}\varphi^2+\mathrm{d}z^2}$$

and, therefore, the components $e_{i'j'}$ of the Euclidean metric in cylindrical coordinates $\{r, \varphi, z\}$ can be read off as

$$\left(e_{i'j'}\right) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{diag}\left(1, r^2, 1\right). \tag{5.6}$$

Example 5.3 Consider the surface of a sphere of radius R and centre in the origin of $\mathbb{R}^3$. We want to compute the metric induced by $\mathbb{R}^3$ on this 2-dimensional sphere using spherical coordinates $\{\theta, \varphi\}$. The coordinate transformation $\{x, y, z\} \longrightarrow \{r, \theta, \varphi\}$ has inverse

$$\begin{aligned} x &= r \sin\theta \cos\varphi, \\ y &= r \sin\theta \sin\varphi, \\ z &= r \cos\theta, \end{aligned}$$

with $r = R =$ const. on the surface of the given sphere. The Euclidean 3-dimensional metric in Cartesian coordinates has components $e_{ij} = \mathrm{diag}\,(1, 1, 1)$ and produces the line element between two infinitesimally nearby points

$$\mathrm{d}l^2_{(3)} = \delta_{ij}\,\mathrm{d}x^i \mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

Now express the differentials $\mathrm{d}x^i = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ on the surface of the sphere in terms of the differentials of θ and φ:

$$\begin{aligned} \mathrm{d}x &= R\left(\cos\theta \cos\varphi\,\mathrm{d}\theta - \sin\theta \sin\varphi\,\mathrm{d}\varphi\right), \\ \mathrm{d}y &= R\left(\cos\theta \sin\varphi\,\mathrm{d}\theta + \sin\theta \cos\varphi\,\mathrm{d}\varphi\right), \\ \mathrm{d}z &= -R\,\sin\theta\,\mathrm{d}\theta, \end{aligned}$$

hence,

$$\begin{aligned}
\mathrm{d}l_{(3)}^2 &= R^2 \Big[(\cos\theta\cos\varphi\,\mathrm{d}\theta - \sin\theta\sin\varphi\,\mathrm{d}\varphi)^2 \\
&\quad + (\cos\theta\sin\varphi\,\mathrm{d}\theta + \sin\theta\cos\varphi\,\mathrm{d}\varphi)^2 + \sin^2\theta\,\mathrm{d}\theta^2 \Big] \\
&= R^2 \Big[\cos^2\theta \left(\cos^2\varphi + \sin^2\varphi\right) \mathrm{d}\theta^2 + \sin^2\theta \left(\sin^2\varphi + \cos^2\varphi\right) \mathrm{d}\varphi^2 \\
&\quad + \sin^2\theta d\theta^2 \Big] = R^2 \left(d\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \right).
\end{aligned}$$

By calling $e_{ij}^{(s)}$ the restriction of the Euclidean metric e_{ij} to the 2-sphere of radius R, we can write the line element on this 2-sphere as

$$\mathrm{d}l_{(2)}^2 = e_{i'j'}^{(s)}\,\mathrm{d}x^{i'}\mathrm{d}x^{j'} = R^2 d\theta^2 + R^2\sin^2\theta\,\mathrm{d}\varphi^2 \qquad \left(i', j' = 1, 2\right), \tag{5.7}$$

or

$$\left(e_{i'j'}^{(s)}\right) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{5.8}$$

in coordinates $\{\theta, \varphi\}$. A widely used notation for the line element on the unit 2-sphere is

$$\mathrm{d}\Omega_{(2)}^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2. \tag{5.9}$$

* * *

In Cartesian coordinates, the inverse Minkowski metric $\eta^{\mu\nu}$ is equal to $\eta_{\mu\nu}$:

$$\eta^{\mu\nu} \doteq \mathrm{diag}\,(-1, 1, 1, 1). \tag{5.10}$$

Let us update our definition of Minkowski spacetime: the *Minkowski spacetime* is the pair $\left(\mathbb{R}^4, \eta_{\mu\nu}\right)$, the set of all spacetime events endowed with the Minkowski metric $\eta_{\mu\nu}$. This definition reflects the fact that it is not only the set of events $\mathbb{R}^4$ that matters, but also the metric with which it is endowed. The relations between spacetime points determined by this metric are more important than the events themselves.

The *Minkowski metric in cylindrical coordinates* $\{ct, r, \theta, z\}$ is given by

$$ds^2 \doteq -c^2dt^2 + dr^2 + r^2d\theta^2 + dz^2, \tag{5.11}$$

or

$$g_{\mu\nu} \doteq \mathrm{diag}\left(-1, 1, r^2, 1\right) \tag{5.12}$$

with

$$g^{\mu\nu} \doteq \mathrm{diag}\left(-1, 1, 1/r^2, 1\right). \tag{5.13}$$

The *Minkowski metric in spherical coordinates* $\{ct, r, \theta, \varphi\}$ is given by

$$ds^2 \doteq -c^2dt^2 + dr^2 + r^2d\Omega^2_{(2)}, \tag{5.14}$$

or

$$g_{\mu\nu} \doteq \mathrm{diag}\left(-1, 1, r^2, r^2\sin^2\theta\right), \tag{5.15}$$

$$g^{\mu\nu} \doteq \mathrm{diag}\left(-1, 1, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right). \tag{5.16}$$

In Minkowski spacetime the Minkowski metric $\eta_{\mu\nu}$ satisfies the requirement of non-singularity and the inverse metric $\eta^{\mu\nu}$ is well-defined. In Cartesian coordinates $\{ct, x, y, z\}$ we have $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1) = \eta^{\mu\nu}$,

$$\mathrm{Det}\left(\eta_{\mu\nu}\right) = -1,$$

and

$$\eta_{\mu\nu}\,\eta^{\nu\alpha} = \delta^\alpha_\mu.$$

5.4 Scalar Product and Length of a Vector in Minkowski Spacetime

In addition to the distance between spacetime points, the metric tensor provides the notion of scalar product between 4-vectors and that of length of a 4-vector in Minkowski spacetime. Remember that the scalar product between two vectors X^α and Y^β is $< \mathsf{X}, \mathsf{Y} > \equiv g_{\alpha\beta}\, X^\alpha Y^\beta$ and that the length squared of a vector X^α is the scalar product of X^α with itself $< \mathsf{X}, \mathsf{X} > \equiv g_{\alpha\beta}\, X^\alpha X^\beta$. In Minkowski spacetime, the length of a vector is not positive-definite because of the Lorentzian signature of the Minkowski metric. In Cartesian coordinates we have

$$\eta_{\mu\nu} X^\mu X^\nu = -\left(X^0\right)^2 + \left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2.$$

This means that a vector can have zero length even if its components are not all zero. For example, the vectors with components

$$l^\mu = \left(l^0, \pm l^0, 0, 0\right) \qquad (l^0 \neq 0)$$

in Cartesian coordinates have length

$$\eta_{\mu\nu}\, l^\mu\, l^\nu = -\left(l^0\right)^2 + \left(l^0\right)^2 = 0.$$

These vectors are orthogonal to themselves but are not identically zero. Again, two vectors X^α, Y^β are orthogonal if $g_{\alpha\beta} X^\alpha Y^\beta = 0$.

Definition 5.1 If two vectors X^α, Y^β have non-zero lengths, the *angle θ between X^α and Y^β* is defined by

$$\cos\theta \equiv \frac{g_{\mu\nu} X^\mu Y^\nu}{\sqrt{|g_{\alpha\beta} X^\alpha X^\beta| \cdot |g_{\rho\sigma} Y^\rho Y^\sigma|}}. \tag{5.17}$$

Example 5.4 In $\mathbb{R}^3$, the scalar product defined by the Euclidean metric e_{ij} coincides with the ordinary dot product of vectors. In Cartesian coordinates, for two vectors $\mathbf{a} = \left(a^x, a^y, a^z\right)$, $\mathbf{b} = \left(b^x, b^y, b^z\right)$,

$$e_{ij}\, a^i b^j = \delta_{ij} a^i b^j = a^x b^x + a^y b^y + a^z b^z \equiv \mathbf{a} \cdot \mathbf{b}; \tag{5.18}$$

the length squared of a vector **a** is

$$e_{ij}\, a^i a^j = \delta_{ij}\, a^i a^j = a^x a^x + a^y a^y + a^z a^z = \mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2. \tag{5.19}$$

In cylindrical coordinates $\{r, \varphi, z\}$, using the expression (5.12) we obtain

$$e_{i'j'}\, a^{i'} a^{j'} = \begin{pmatrix} a^r \\ a^\varphi \\ a^z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^r \\ a^\varphi \\ a^z \end{pmatrix} = \left(a^r\right)^2 + r^2 \left(a^\varphi\right)^2 + \left(a^z\right)^2.$$

* * *

In the space $\mathbb{R}^n$ *with Euclidean metric* and *in Cartesian coordinates*, vector components with upper and lower indices are the same because the metric reduces to the Kronecker delta.[2] For example,

[2] This is the reason why, so far, we have not distinguished between contravariant components A^i and covariant components A_i of a vector $\mathbf{A}$ in $\mathbb{R}^3$ with Cartesian coordinates.

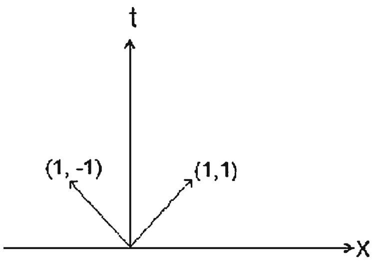

Fig. 5.1 The null vectors $(1, \pm 1)$ defining a cone (*light cone*) in 2-dimensional Minkowski spacetime

$$A_i = g_{ij} A^j = \delta_{ij} A^j = A^i,$$

$$A^k = g^{kl} A_l = \delta^{kl} A_l = A_k,$$

$$T_{ij} = g_{il}\, g_{jm}\, T^{lm} = \delta_{il}\, \delta_{jm}\, T^{lm} = T^{ij}.$$

Of course, this property is no longer true when non-Cartesian coordinates are used and $g_{ij} \neq \delta_{ij}$, or when a Lorentzian metric is used instead of the Euclidean one.

Example 5.5 Consider the 2-dimensional Minkowski spacetime $\mathbb{R}^2$ with the Minkowski metric

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

in coordinates $\{ct, x\}$. The vector $X^\mu = (1, 1)$ is not identically zero but

$$\eta_{\mu\nu} X^\mu X^\nu = -(1)^2 + (1)^2 = 0;$$

the same is true for the vector $Y^\mu = (1, -1)$. These two vectors identify the future light cone through the origin of this Minkowski space (Fig. 5.1).

Vectors which appear to have the same length in a (x, t) Minkowski diagram (suppressing two dimensions) do not necessarily have the same length according to the Minkowski metric. Similarly, vectors which appear to be orthogonal in the (x, t) spacetime diagram are not necessarily orthogonal in the sense of the Minkowski metric. Consider, for example $A^\mu = (1, 1, 0, 0)$ and $B^\mu = (1, -1, 0, 0)$; it is

$$A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 = -2 \neq 0$$

but these two vectors appear orthogonal in a Minkowski diagram (Fig. 5.1). The representation in the spacetime diagram distorts four-dimensional lengths and angles. However, the sum of two vectors in the diagram corresponds to their sum in the sense of the Minkowski metric and the graphical notion of parallelism of two vectors (i.e., $A^\mu = \lambda B^\mu$ for some scalar $\lambda > 0$ defines "A^μ is parallel to B^μ") corresponds to parallelism in the sense of the Minkowski metric.

5.5 Raising and Lowering Tensor Indices

As we have seen, the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ can be used to lower or raise tensor indices.

- For any contravariant vector X^μ we can define a corresponding covariant vector $X_\mu \equiv g_{\mu\nu} X^\nu$.
- For any covariant vector Y_α we can define a corresponding contravariant vector $Y^\alpha \equiv g^{\alpha\beta} Y_\beta$.

Similarly, any tensor index can be raised or lowered using $g^{\alpha\beta}$ or $g_{\alpha\beta}$. For example, we can associate to the tensor $T^\alpha{}_\beta$ both $T_{\alpha\beta} \equiv g_{\alpha\gamma} T^\gamma{}_\beta$ and $T^{\alpha\beta} \equiv g^{\beta\gamma} T^\alpha{}_\gamma$. If we want to lower the index α_j of the tensor $T^{\alpha_1 \dots \alpha_j \dots \alpha_k}{}_{\beta_1 \dots \beta_l}$, we use $g_{\alpha_j\gamma}$ obtaining

$$T^{\alpha_1 \dots \alpha_{j-1}\alpha_{j+1} \dots \alpha_n}{}_{\alpha_j \beta_1 \dots \beta_l} \equiv g_{\alpha_j \gamma}\, T^{\alpha_1 \dots \alpha_{j-1} \gamma\, \alpha_{j+1} \dots \alpha_k}{}_{\beta_1 \dots \beta_l} . \tag{5.20}$$

X^α and X_α are two different representations of the same vector provided by $g_{\alpha\beta}$, $g^{\alpha\beta}$, and δ^α_β.

In Minkowski spacetime a contravariant 4-vector in Cartesian coordinates is written as

$$A^\mu = \left(A^0, \mathbf{A}\right) \equiv \left(A^0, A^x, A^y, A^z\right)$$

and the corresponding covariant 4-vector (dual vector) is

$$A_\mu = \eta_{\mu\nu} A^\nu = \left(-A^0, \mathbf{A}\right) = \left(-A^0, A^x, A^y, A^z\right) = \left(A_0, A_x, A_y, A_z\right) .$$

Example 5.6 In Minkowski spacetime in Cartesian coordinates, find the components T_{00} and $T_0{}^0$ of a tensor $T^{\mu\nu}$.
We have

$$T_{00} = \eta_{0\mu}\, \eta_{0\nu}\, T^{\mu\nu} \underset{\substack{\uparrow \\ \text{in} \\ \text{Cartesian} \\ \text{coordinates}}}{\doteq} \left(-\delta_{0\mu}\right) \left(-\delta_{0\nu}\right) T^{\mu\nu} = T^{00}$$

and

$$T_0{}^\nu = \eta_{0\mu} T^{\mu\nu} = -\delta_{0\mu} T^{\mu\nu} = -T^{0\nu} ;$$

in particular, for $\nu = 0$ one has $T_0{}^0 = -T^{00}$.

* * *

The scalar product can also be used to define the divergence of a vector field **A** in the 3-dimensional space with Euclidean metric and in Cartesian coordinates:

$$\nabla \cdot \mathbf{A} \equiv g^{ij} \frac{\partial A_i}{\partial x^j} = \delta^{ij} \frac{\partial A_i}{\partial x^j} = \frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3},$$

which matches the familiar expression

$$\nabla \cdot \mathbf{A} \equiv \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z}.$$

Definition 5.2 The *divergence of a 4-vector field* $A^\mu = \left(A^0, \mathbf{A}\right)$ in Minkowski spacetime in Cartesian coordinates is

$$\begin{aligned} \partial_\mu A^\mu &= \delta^\mu_\nu \partial_\mu A^\nu = \delta^{\mu\nu} \partial_\mu A_\nu \\ &= \frac{\partial A^0}{\partial\,(ct)} + \frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} + \frac{\partial A^z}{\partial z} \end{aligned} \tag{5.21}$$

so

$$\partial_\mu A^\mu = \frac{\partial A^0}{\partial (ct)} + \nabla \cdot \mathbf{A}. \tag{5.22}$$

Definition 5.3 The *d'Alembertian* of a scalar field ϕ is

$$\Box\phi \doteq\equiv \partial^\mu \partial_\mu \phi \doteq \eta^{\mu\nu} \partial_\mu \partial_\nu \phi \tag{5.23}$$

in Cartesian coordinates,[3] for which

$$\Box\phi \doteq \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = -\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi, \tag{5.24}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the usual Laplace operator in Cartesian coordinates.

[3] The definition of the d'Alembertian $\nabla^\mu \nabla_\mu \phi$ in general coordinates requires the notion of covariant derivative ∇_α introduced in Chap. 10.

The d'Alembertian used in wave mechanics (when the waves propagate at the speed of light c) appears to be a straightforward generalization of the Laplacian to four dimensions with the Lorentzian signature.

The d'Alembertian of ϕ coincides with the divergence of the gradient $\partial_\mu \phi$, or $\Box\phi \doteq \partial^\mu \left(\partial_\mu \phi\right)$. This is analogous to the situation in three dimensions in which the Laplacian is the divergence of the gradient: $\nabla^2\phi = \nabla \cdot (\nabla\phi)$.

5.5.1 Working with Tensors in Minkowski Spacetime

From a 4-vector $A^\mu = \left(A^0, \mathbf{A}\right) = \left(A^0, A^i\right)$ $(i = 1, 2, 3)$ one can obtain covariant components by lowering the indices with the Minkowski metric. In Cartesian coordinates $\left\{x^\mu\right\} = \{ct, x, y, z\}$, it is

$$A_\mu = \eta_{\mu\nu}\, A^\nu = (A_0, A_i) = \left(-A^0, A^i\right) . \tag{5.25}$$

If A^μ, B^μ are two 4-vectors then

$$A_\mu B^\mu = \eta_{\mu\nu}\, A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 = -A^0 B^0 + \mathbf{A} \cdot \mathbf{B}.$$

We have also

$$A_\mu A^\mu = -\left(A^0\right)^2 + (\mathbf{A})^2 .$$

The transformation property of the components of a 4-vector A^μ under Lorentz transformations in standard configuration is $A^\mu \to A^{\mu'}$ with

$$A^{0'} = \gamma\left(A^0 - \frac{v}{c}\, A^1\right), \tag{5.26}$$

$$A^{1'} = \gamma\left(A^1 - \frac{v}{c}\, A^0\right), \tag{5.27}$$

$$A^{2'} = A^2, \tag{5.28}$$

$$A^{3'} = A^3, \tag{5.29}$$

as follows from $A^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\alpha}\, A^\alpha \equiv L_{(v)}{}^{\mu'}{}_\alpha\, A^\alpha$. For a covariant vector B_μ, we have

$$B_{0'} = \gamma\left(B_0 + \frac{v}{c}\, B_1\right), \tag{5.30}$$

$$B_{1'} = \gamma\left(B_1 + \frac{v}{c}\, B_0\right), \tag{5.31}$$

$$B_{2'} = B_2, \tag{5.32}$$

$$B_{3'} = B_3, \tag{5.33}$$

according to $B_{\mu'} = \dfrac{\partial x^\alpha}{\partial x^{\mu'}} B_\alpha = L_{(v)\,\mu'}{}^{\alpha} B_\alpha$.

For higher rank tensors, one applies repeatedly the Lorentz matrix $L_\alpha{}^{\beta'} = \dfrac{\partial x^{\beta'}}{\partial x^\alpha}$ and its inverse, for example

$$T^{\alpha'\beta'}{}_{\gamma'} = L^{\alpha'}{}_{\rho}\, L^{\beta'}{}_{\sigma}\, L^{\delta}{}_{\gamma'}\, T^{\rho\sigma}{}_{\delta}.$$

5.6 Causal Nature of 4-Vectors

A vector X^μ in Minkowski spacetime is

- *timelike* if $X_\mu X^\mu < 0$,
- *null* or *lightlike* if $X_\mu X^\mu = 0$,
- *spacelike* if $X_\mu X^\mu > 0$.

A timelike or null vector is called a *causal vector* (Fig. 5.3).

The *light cone* (or *null cone*) at a spacetime point P is the set of null vectors at P. This is a vector space of dimension 2 and a surface in Minkowski space (and, as will be clear later, is generated by the tangents to ingoing and outgoing radial null rays at P).

In Cartesian coordinates, a null vector satisfies

$$\eta_{\mu\nu}\, X^\mu X^\nu \doteq 0$$

or

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 = 0.$$

If X^μ coincides with the Cartesian position 4-vector x^μ, this is the equation of the double cone

$$x^0 = \pm\sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2}$$

(Fig. 5.2). Let $t^\mu = (1, 0, 0, 0) = \delta^{0\mu}$; this unit vector points in the "direction of time" (the direction of the time axis) and has unit norm.

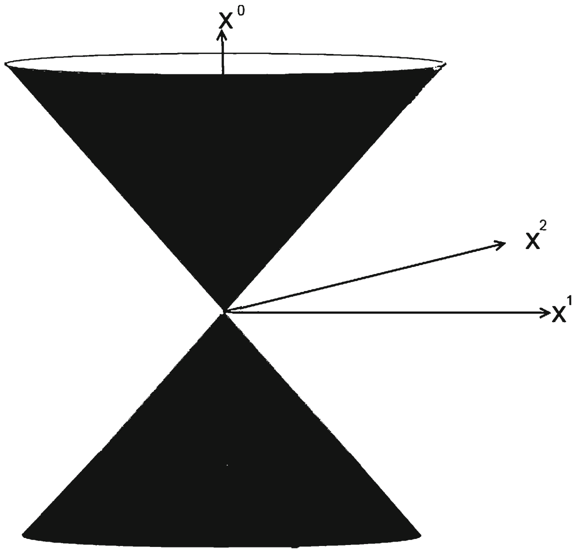

Fig. 5.2 The light cone through the spacetime point $O = (0, 0, 0, 0)$

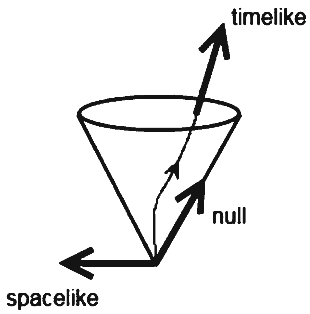

Fig. 5.3 A timelike vector at a spacetime point P points *inside* the light cone at P; a null vector points *along* the light cone, and a spacelike vector points *outside* of it. The tangent to the worldline of a massive particle always points inside the light cone

A timelike or null vector X^μ is

- *future-pointing* if $t_\mu X^\mu < 0$, or
- *past-pointing* if $t_\mu X^\mu > 0$.

Note that this definition involves a scalar product, therefore it is independent of the coordinate system used.

Let X^μ be a timelike or null vector and let $\left(X^0, X^1, X^2, X^3\right)$ be its components in a coordinate system $\left\{x^\mu\right\} = \left\{x^0, \mathbf{x}\right\}$, with time (multiplied by c) as the zeroth component. If $X^0 > 0$, then X^μ is future-pointing, while if $X^0 < 0$, X^μ is past-pointing, and $\eta_{\mu\nu} t^\mu X^\nu \equiv t_\mu X^\mu$ is the projection of X^α on the time direction.

Two causal vectors are called *isochronous* if they are both future-pointing or both past-pointing. This means that, if $A^\mu = \left(A^0, \mathbf{A}\right)$ and $B^\mu = \left(B^0, \mathbf{B}\right)$, it is $A^0 B^0 > 0$.

Example 5.7 The 4-vector of components $A^\mu = (1, 0, 3, 1)$ is spacelike since

$$A_\mu A^\mu = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 = -1 + 9 + 1 = 9 > 0.$$

The 4-vector $l^\mu = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ is null since

$$l_\mu l^\mu = -(l^0)^2 + (l^1)^2 + (l^2)^2 + (l^3)^2 = -\frac{1}{4} + \frac{1}{4} = 0.$$

The 4-vector $B^\mu = (3, 1, 1, 0)$ is timelike since

$$B_\mu B^\mu = -(B^0)^2 + (B^1)^2 + (B^2)^2 + (B^3)^2 = -9 + 1 + 1 = -7 < 0.$$

* * *

It is often convenient to choose an inertial frame which simplifies the calculations. The following results are useful to this regard:

- if $A^\mu = \left(A^0, \mathbf{A}\right)$ is a timelike 4-vector, it is always possible to find an inertial frame in which $A^{\mu'} = \left(A^{0'}, 0, 0, 0\right)$. This frame is unique.
- For a *spacelike* 4-vector B^μ, it is always possible to find an inertial frame in which the components reduce to $B^{\mu'} = \left(0, B^{1'}, 0, 0\right)$. This frame is unique.
- (*Zero component lemma*) If a 4-vector has the same component (for example the time component) equal to zero in *all* inertial frames, it must be the zero vector $(0, 0, 0, 0)$.

The proof of these statements is left as an exercise.

In a Minkowski diagram, the frame S has its time axis vertical and its x-axis horizontal and simultaneities (events occurring at the same time in this frame) form a horizontal straight line. All other inertial frames have apparently non-orthogonal t'- and x'- axes and simultaneities of these other frames are represented by oblique lines in the (x, t) plane. The apparent orthogonality has no physical significance because the physical metric is the Minkowski one, not the Euclidean metric upon which our intuition is built and which suggests orthogonality in the Euclidean sense. Taking different inertial frames corresponds to taking different time slices of Minkowski spacetime (which have different time axes) and with hyperplanes inclined with respect to the 3-spaces t =const. of S (Fig. 5.4).

A null vector $l^\mu = \left(l^0, \mathbf{l}\right)$ can always be reduced to the form

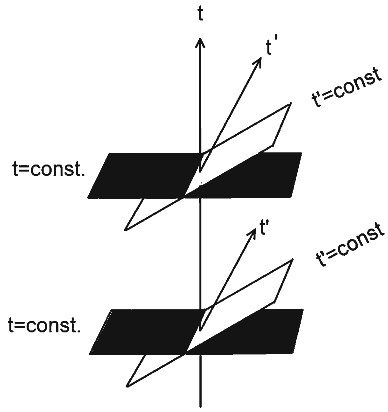

Fig. 5.4 Two slicings of Minkowski spacetime with two different times and 3-dimensional spaces

$$l^{\mu'} = \left(l^{0'}, \pm l^{0'}, 0, 0\right)$$

(this is trivial to show: simply align the x'-axis with $\pm\mathbf{l}$). Note also that a null vector is defined up to a constant without affecting its normalization. If l^μ is such that $l^\mu l_\mu = 0$, then $\forall\,\alpha \neq 0$, $m^\mu = \alpha\, l^\mu$ is parallel to l^μ and null: $m_\mu m^\mu = \alpha^2\, l^\mu l_\mu = 0$. If l^μ is future- or past-pointing one must choose the constant α positive in order for m^μ to remain future- or past-pointing.

The following statements hold true, assuming that none of the null 4-vectors involved coincide with the trivial vector (0, 0, 0, 0):

- The sum of two isochronous timelike 4-vectors is a timelike 4-vector isochronous with them.
- The sum of a timelike 4-vector and an isochronous null 4-vector is a timelike 4-vector isochronous with both.
- The sum of two isochronous null 4-vectors is a timelike 4-vector unless the two 4-vectors are parallel, in which case their sum is a null 4-vector.
- The difference of two isochronous null 4-vectors is a spacelike 4-vector unless the two null 4-vectors are parallel, in which case their difference is a null 4-vector.
- The sum of any number of isochronous null or timelike 4-vectors is a timelike or null 4-vector isochronous with them and it is null if and only if all the 4-vectors added are null and parallel.
- Any timelike 4-vector can be expressed as the sum of two isochronous null 4-vectors.
- Any spacelike 4-vector can be expressed as the difference of two isochronous null 4-vectors.
- A timelike 4-vector cannot be orthogonal to a causal 4-vector.
- A 4-vector orthogonal to a null 4-vector A^μ must be spacelike, or else it is null and parallel or antiparallel to A^μ.
- Any 4-vector orthogonal to a causal 4-vector A^μ is spacelike, or else it is a null 4-vector parallel to A^μ.

- The scalar product of two isochronous timelike 4-vectors is negative.
- The scalar product of two isochronous null 4-vectors is negative unless they are parallel (in which case their product vanishes).
- The scalar product of a timelike 4-vector and an isochronous null 4-vector is negative.

The proofs of these statements are left as exercises.

Definition 5.4 A curve $x^\mu(\lambda)$ in Minkowski spacetime is a timelike/null/spacelike curve at a point if its 4-tangent $u^\mu = dx^\mu/d\lambda$ is timelike/null/spacelike, respectively, at that point. A *timelike curve* is one whose 4-tangent is everywhere timelike, etc., i.e., the causal character of a spacetime curve is the causal character of its tangent.

The worldline of a massive particle is a timelike curve, while a null ray (the spacetime trajectory of a photon) is a null curve.

Example 5.8 Consider the geometric curve $x^\mu(\lambda) = \left(3\lambda, 6\lambda^2, 0, 0\right)$ in Minkowski spacetime in Cartesian coordinates. The 4-tangent to this curve is

$$u^\mu = \frac{dx^\mu}{d\lambda} = (3, 12\lambda, 0, 0)$$

and its square is

$$u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 = -9 + 144\lambda^2.$$

We have $u_\mu u^\mu < 0$ if $|\lambda| < 1/4$ and the curve is spacelike for $\lambda < -1/4$, null at $\lambda = 1/4$, timelike for $-1/4 < \lambda < 1/4$, null again at $\lambda = 1/4$, and then spacelike again for $\lambda > 1/4$ (this curve cannot be the worldline of a physical particle).

Example 5.9 Consider the curve with parametric representation

$$x^\mu(\lambda) = \frac{1}{\sqrt{7}}\left(3\lambda, \lambda, 3, \lambda\right)$$

in Minkowski spacetime in Cartesian coordinates. The 4-tangent is

$$u^\mu = \frac{dx^\mu}{d\lambda} = \frac{1}{\sqrt{7}}\left(3, 1, 0, 1\right)$$

and its square is $u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -9 + 1 + 1 = -1 < 0$. This curve is timelike.

Example 5.10 Consider the curve of parametric representation

$$x^\mu(\lambda) = \left(3\lambda^2, 1, 13, 3\lambda^2\right)$$

in Minkowski spacetime in Cartesian coordinates. The 4-tangent is $u^\mu = \frac{dx^\mu}{d\lambda} = (6\lambda, 0, 0, 6\lambda)$ and its square is $u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -36\lambda^2 + 36\lambda^2 = 0$. This curve is always null (and, therefore, it could represent the worldline of a photon).

5.7 Hypersurfaces

Definition 5.5 A *hypersurface* in an n-dimensional space is a surface of dimension $n - 1$. A hypersurface is

- *timelike* if its normal n^μ is spacelike, $n_\mu n^\mu > 0$;
- *null* if its normal n^μ is null, $n_\mu n^\mu = 0$;
- *spacelike* if its normal n^μ is timelike, $n_\mu n^\mu < 0$.

Example 5.11 Any hypersurface t =constant is spacelike. In fact, the equation of the hypersurface is

$$f(t) = t - \text{const.} = 0.$$

The normal to this surface has the direction of the gradient of f,

$$n_\mu = \nabla_\mu f = \nabla_\mu t = (1, 0, 0, 0)$$

and is already normalized:

$$n_\mu n^\mu = -1.$$

Example 5.12 Any hypersurface x^1 = constant is timelike. In fact, the equation of this surface is $f(x) = x^1 -$const. $= 0$. The normal has the direction of the gradient,

$$n_\mu = \nabla_\mu f = (0, 1, 0, 0)$$

and is normalized, $n_\mu n^\mu = 1$ and spacelike, hence $x^1 =$ const. is a timelike hypersurface.

* * *

The null cone through any point of Minkowski spacetime is a null surface.

Proof Let $x^\mu_{(0)} = (ct_0, x_0, y_0, z_0)$ in Cartesian coordinates (the result, however, will not depend on the coordinates adopted). The light cone through $x^\mu_{(0)}$ has equation

$$f(t, x, y, z) \equiv -c^2 (t - t_0)^2 + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 0.$$

The normal to this surface is

$$n_\mu = \nabla_\mu f \Big|_{f=0}$$

$$= -2c\,(t-t_0)\,\delta_{0\mu} + 2\,(x-x_0)\,\delta_{1\mu} + 2\,(y-y_0)\,\delta_{2\mu}$$

$$+2\,(z-z_0)\,\delta_{3\mu}\Big|_{f=0}$$

or

$$n_\mu = 2\Big(-c\,(t-t_0)\,, x-x_0, y-y_0, z-z_0\Big),$$

while

$$n^\mu = 2\Big(c\,(t-t_0)\,, x-x_0, y-y_0, z-z_0\Big)$$

so that

$$n^\mu n_\mu = 4\Big[-c^2\,(t-t_0)^2 + (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\Big]_{f=0}$$
$$= 4f\,(t,x,y,z)\Big|_{f=0} = 0\,.$$

□

The null cone is a 2-dimensional surface.

Proof The null cone through any point P of Minkowski spacetime is generated by only two linearly independent null vectors at that point. If $l^\mu = \left(l^0, \mathbf{l}\right)$ is a null vector at P, one can align the x-axis with $\mathbf{l}$ and then, in these coordinates, it is $l^\mu = \left(l^0, l^1, 0, 0\right)$. The normalization $l_\mu l^\mu = -(l^0)^2 + (l^1)^2 = 0$ yields $l^1 = \pm l^0$ and $l^\mu_{(1,2)} = \left(l^0, \pm l^0, 0, 0\right)$. These are all the null vectors at P. It is easy to see that $l^\mu_{(1)}$ and $l^\mu_{(2)}$ are linearly independent and, therefore, the null cone generated by them has dimension two.

5.8 Gauss' Theorem

Let Ω be a 4-dimensional simply connected region of Minkowski spacetime which has volume element $d\Omega \equiv \sqrt{|g|}\,\mathrm{d}^4x \doteq \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ in Cartesian coordinates in which $g \doteq \mathrm{Det}\left(\eta_{\mu\nu}\right) = -1$. Let $\partial\Omega \equiv S$ be its 3-dimensional boundary with its own 3-dimensional volume element ("surface element") $\mathrm{d}S$ associated with an outward-pointing normal n^μ. This normal is assumed to be normalized, i.e., it has norm squared $+1$ if the surface is timelike, -1 if the surface is spacelike and, of course, zero if the surface is null. Let V^μ be a 4-vector field defined in Ω. The *Gauss theorem* states that

$$\int_\Omega d^4x\, \partial_\mu V^\mu = \int_{\partial\Omega} dS\, V^\mu n_\mu . \tag{5.34}$$

Example 5.13 Consider the spacetime region

$$\Omega = \{(ct, r, \vartheta, \varphi) : \quad 0 \le r \le r_0\} ,$$

which is a"tube" formed by the region of 3-dimensional space enclosed by the sphere of radius r_0, with time spanning the entire t-axis. The boundary of this regions is

$$\partial\Omega = \{(ct, r, \vartheta, \varphi) : \quad r = r_0\} ,$$

and the unit normal to $\partial\Omega$ is

$$n^\mu = (0, 1, 0, 0) , \qquad n_\mu = (0, 1, 0, 0) .$$

Consider the vector field

$$V^\mu = \left(0, \frac{r}{1+\alpha t^2}, 0, 0, \right) = \left(V^0, \mathbf{V}\right) ,$$

where $\alpha > 0$ is a constant with the dimensions of an inverse time squared. The 4-divergence of V^μ is

$$\partial_\mu V^\mu = \frac{\partial V^0}{\partial(ct)} + \nabla \cdot \mathbf{V} = 0 + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V^r\right) + 0 + 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{1+\alpha t^2}\right) = \frac{3}{1+\alpha t^2}$$

and its integral over the 4-region Ω is

$$\int_\Omega d^4x\, \partial_\mu V^\mu = \int_\Omega d^4x\, \frac{3}{1+\alpha t^2} = \int_{-\infty}^{+\infty} dt \int_0^{r_0} dr \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi\, r^2 \sin\vartheta\, \frac{3}{1+\alpha t^2}$$
$$= 3\,\frac{4\pi}{3}\, r_0^3 \int_{-\infty}^{+\infty} \frac{dt}{1+\alpha t^2} = 4\pi r_0^3 \left[\frac{1}{\sqrt{\alpha}} \tan^{-1}\left(\sqrt{\alpha}\, t\right)\right]_{-\infty}^{+\infty} = \frac{4\pi^2 r_0^3}{\sqrt{\alpha}} .$$

Now compute the surface integral of the vector field V^μ itself on the boundary $\partial\Omega$:

$$\int_{\partial\Omega} dS\, n_\mu V^\mu = \int_{-\infty}^{+\infty} dt \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi\, r^2 \sin\vartheta\, \frac{r}{1+\alpha t^2}\bigg|_{r=r_0}$$

$$= 4\pi r_0^3 \int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{1}{1 + \alpha t^2} = \frac{4\pi^2 r_0^3}{\sqrt{\alpha}} \, .$$

Therefore, Gauss' theorem $\int_{\Omega} d^4x \, \partial_\mu V^\mu = \int_{\partial\Omega} \mathrm{d}S \, n_\mu V^\mu$ is verified.

5.9 Conclusion

Thus far, we have seen that a 4-dimensional world view is convenient and even necessary because a change of inertial frame mixes space and time coordinates, similar to the way in which spatial rotations in three dimensions mix different spatial coordinates. We have studied the geometry of Minkowski spacetime and it is now time to do physics in this spacetime. Physics must be given a relativistic (i.e., Lorentz-invariant) formulation. Beginning with mechanics, we know that Newton's second law is Galilei- but not Lorentz-invariant, and it must be modified. Maxwell's theory is already Lorentz-invariant and does not need to be modified, but only rewritten in the 4-dimensional formalism.

A physical theory will be expressed by basic physical laws which must be theoretically and experimentally consistent with our (limited) knowledge and must make new predictions which are falsifiable.[4] Further, these laws must be expressed in a covariant way by tensor equations. We will only consider physics without gravity (gravity is included in General Relativity but not in Special Relativity) and with the stipulation that there exists a preferred class of reference frames, the inertial frames. We begin our study of physics in Minkowski spacetime by reformulating the mechanics of point particles (we know that Newtonian mechanics is not invariant under Lorentz boosts) and then moving on to geometric optics, fluid physics, and the physics of scalar and electromagnetic fields, while giving some general prescriptions applicable to any branch of physics whenever possible.

The relativistic corrections to Newtonian mechanics and their predictions were studied theoretically long before their experimental verification. Today, relativistic mechanics is the basis for the working of particle physics accelerators, nuclear power generation, the Global Positioning System, positron annihilation spectroscopy, and various tools used in medicine and the industry. Newtonian mechanics is adequate in the limit of small velocities $|v| \ll c$. In particle physics experiments, instead, γ-factors of order 10^4 have been achieved and factors $\gamma \sim 10^{11}$ are common in cosmic rays.

[4] It is an old Popperian adage that a theory cannot be verified: it can only be falsified.

Problems

5.1 Find all the future-oriented and all the past-oriented null vectors of the 2-dimensional Minkowski spacetime (ct, x) with the Minkowski metric

$$\eta_{\mu\nu} = \text{diag}\,(-1, 1)\,.$$

5.2 Let A^μ be a 4-vector in Minkowski spacetime. Prove directly, using the transformation properties, that $g_{\mu\nu}A^\mu A^\nu$ is invariant under arbitrary coordinate transformations $x^\mu \longrightarrow x^{\mu'}$.

5.3 Are the following 4-vectors (with components given in Cartesian coordinates) orthogonal to each other in Minkowski spacetime?

$$\begin{aligned} A^\mu &= (1, 0, 0, 1)\,, \\ B^\mu &= (1, 0, 0, 0)\,, \\ C^\mu &= (0, 1, 0, 0)\,. \end{aligned}$$

5.4 Determine the timelike, spacelike, or null character of the 4-vectors

$$\begin{aligned} u^\mu &= (1, 0, 0, 0)\,, & v^\mu &= (1, 1, 1, 1)\,, \\ w^\mu &= (1, 0, 0, 1)\,, & x^\mu &= \left(1, 0, 3, \sqrt{3}\right), \\ y^\mu &= (1, 0, -1, 1)\,, & z^\mu &= (100, 3, 4, 17)\,, \\ q^\mu &= (0, 1, 0, 0)\,, & t^\mu &= \left(5, 0, 0, \sqrt{7}\right), \\ r^\mu &= (0, 1, 0, 1)\,, & s^\mu &= \left(\sqrt{5}, \pi, \sqrt{11}, \mathrm{e}\right), \end{aligned}$$

in Minkowski spacetime in Cartesian coordinates.

5.5 Determine the spacelike, null, or timelike character of the 4-vectors given, in Cartesian coordinates, by

$$\begin{aligned} A^\mu &= \left(1, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ B^\mu &= \left(1, 0, \frac{1}{2}, \frac{1}{2}\right), \\ C^\mu &= (0, 0, 1, 0)\,, \\ D^\nu &= A^{\mu\nu}E_\mu, \quad \text{where } \mathrm{A}^{\mu\nu} = \delta^{\mu 0}\delta^{\nu 1} \text{ and } \mathrm{E}^\mu = \left(\frac{1}{2}, 1, 0, 1\right). \end{aligned}$$

5.6 If u^μ is timelike and s^μ is spacelike, is it true that $u_\mu v^\mu = 0$?

5.7 Prove the following statements or disprove them with counterexamples. In general,

(a) is the sum of two null 4-vectors a null 4-vector?
(b) Is the sum of two spacelike 4-vectors a spacelike 4-vector?
(c) Is the sum of two timelike 4-vectors a timelike 4-vector?

5.8 (a) Let X^μ and Y^μ be two spacelike 4-vectors in Minkowski spacetime. Is $X_\mu Y^\mu \geq 0$?
(b) Let X^μ and Y^μ be two timelike 4-vectors in Minkowski spacetime. Is $X_\mu Y^\mu \leq 0$?
(c) Let X^μ and Y^μ be a timelike and a spacelike 4-vector, respectively. Is $X_\mu Y^\mu = 0$?

5.9 Show that if A^μ is a timelike 4-vector, it is always possible to find a unique inertial frame in which $A^{\mu'} = \left(A^{0'}, 0, 0, 0\right)$.

5.10 Show that if A^μ is a spacelike 4-vector, it is always possible to find a unique inertial frame in which $A^{\mu'} = \left(0, \mathbf{A}'\right)$.

5.11 Show that, if $l^\mu = \left(l^0, \mathbf{l}\right)$ (in Cartesian coordinates) is a null vector, l^0 has the same sign in all inertial frames.

5.12 Prove the zero-component lemma for 4-vectors in Minkowski spacetime.

5.13 Let $X^\mu = (A, B, C, D)$ be a 4-vector in Minkowski spacetime in Cartesian coordinates. Under what conditions on the constants A, B, C, and D is X^μ null and orthogonal to $Y^\mu = (1, 0, 1, 1)$, $Z^\mu = (0, 1, 2, 0)$, $W^\mu = (0, 0, 1, 0)$ and future-pointing?

5.14 Show that, if $A_{\mu\nu} X^\mu X^\nu = 0$ for all 4-vectors X^μ, then $A_{\mu\nu}$ is antisymmetric.

5.15 Show that the sum of two isochronous timelike 4-vectors is a timelike 4-vector isochronous with them.

5.16 Show that the sum of a timelike 4-vector and an isochronous null (non-trivial) 4-vector is a timelike 4-vector isochronous with both.

5.17 Show that the sum of two isochronous null (non-trivial) 4-vectors is a timelike 4-vector unless the two 4-vectors are parallel, in which case their sum is a null 4-vector.

5.18 Show that the difference of two isochronous null (non-trivial) 4-vectors is a spacelike 4-vector unless the two null 4-vectors are parallel, in which case their difference is a null 4-vector.

5.19 Show that the sum of any number of isochronous null (non-trivial) or timelike 4-vectors is a timelike or null 4-vector isochronous with them and it is null if and only if all the 4-vectors added are null and parallel.

5.20 Show that any timelike 4-vector can be expressed as the sum of two isochronous null 4-vectors.

5.21 Show that any spacelike 4-vector can be expressed as the difference of two isochronous null 4-vectors.

5.22 Show that a timelike 4-vector cannot be orthogonal to a causal non-trivial 4-vector.

5.23 Show that any (non-trivial) 4-vector orthogonal to a (non-trivial) causal 4-vector A^μ is spacelike, or else it is a null 4-vector parallel to A^μ.

5.24 Show that the scalar product of two isochronous timelike 4-vectors is negative.

5.25 Show that the scalar product of two isochronous null (non-trivial) 4-vectors is negative unless they are parallel (in which case their product vanishes).

5.26 Show that the scalar product of a timelike 4-vector and an isochronous null (non-trivial) 4-vector is negative.

5.27 Write the Minkowski metric $g_{\mu\nu}$ using the *null coordinates*

$$u \equiv \frac{ct - x}{\sqrt{2}} \qquad (\textit{retarded time}),$$

$$v \equiv \frac{ct + x}{\sqrt{2}} \qquad (\textit{advanced time}).$$

Compute $g_{\mu\nu}$, $\sqrt{|g|}$, and $g^{\mu\nu}$ in these coordinates. Write down the wave equation $\Box\phi = 0$ for a scalar field $\phi = \phi\,(u, v)$. Compute the normals n_μ and m_μ to the surfaces $u =$ const. and $v =$ const., show that they are null 4-vectors, and compute their scalar product $n^\mu m_\mu$. Draw the surfaces $u =$ const. and $v =$ const. in an (x, t) spacetime diagram: what do these surfaces represent?

5.28 Consider a slicing of Minkowski spacetime with hypersurfaces of constant time Σ_t. On each slice Σ_t, consider a 2-sphere

$$\mathscr{S} = \Big\{ (ct, r, \theta, \varphi) : \quad t = \text{const.}, \quad \text{r} = \text{const.} \Big\}.$$

(a) Show that $\mathcal{S}$ is spacelike (could it be otherwise, considering that $\mathcal{S} \subseteq \Sigma_t$?).
(b) Let s^μ be the outward-directed unit normal to $\mathcal{S}$ in Σ_t and let n^μ be the future-pointing timelike unit normal to Σ_t. What is the causal character of $l^\mu \equiv n^\mu + s^\mu$ and $m^\mu \equiv n^\mu - s^\mu$? Normalize l^μ and m^μ so that $l_\mu m^\mu = -1$.

5.29 Given a 2-index tensor $T_{\mu\nu}$, we say that a vector v^μ is an *eigenvector* of $T_{\mu\nu}$ if there exists a scalar λ (*eigenvalue*) such that $T_{\mu\nu} v^\nu = \lambda v_\mu$.
(a) Find all the eigenvectors of the Minkowski metric $\eta_{\mu\nu}$.
(b) Let $T_{\mu\nu}$ be symmetric; what is the maximum number of independent eigenvectors of $T_{\mu\nu}$ in 4-dimensional Minkowski spacetime?
(c) In general, one cannot diagonalize simultaneously the Minkowski metric and a symmetric 2-tensor with a coordinate transformation in Minkowski spacetime. This fact is linked to the existence of null vectors. Take $T^{\mu\nu} = k^\mu k^\nu$, where $k^\mu \doteq (1, 1, 0, 0)$ in Cartesian coordinates and prove that no Lorentz transformation can diagonalize $T^{\mu\nu}$.
(d) Let $F_{\mu\nu}$ be antisymmetric and let v^μ be an eigenvector of $F_{\mu\nu}$ with eigenvalue λ. What can you say about v^μ and/or λ?

5.30 Show that, given two 4-vectors $A^\mu = \left(A^0, \mathbf{A}\right)$ and $B^\mu = \left(B^0, \mathbf{B}\right)$ in Cartesian coordinates in Minkowski spacetime, the quantity

$$\mathscr{I} \equiv \frac{(A^0 - A^1)(B^0 + B^1)}{(A^0 + A^1)(B^0 - B^1)}$$

is Lorentz-invariant.

5.31 In an (x, t) spacetime diagram, draw
(a) an hypersurface which is asymptotically null as $|x| \longrightarrow +\infty$;
(b) an hypersurface which is null in the far past.

5.32 Find the form of the Minkowski line element in the coordinate system $\{ct, r, \theta, \varphi\}$ related to Cartesian coordinates by

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\varphi,$$
$$y = \sqrt{r^2 + a^2} \sin\theta \sin\varphi,$$
$$z = r \cos\theta.$$

In General Relativity, the spacetime outside a rotating stationary black hole of mass M and angular momentum per unit mass a is given by the Kerr metric (here expressed in Boyer-Lindquist coordinates and in units in which Newton's constant and c are unity [1–4])

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4aMr\sin^2\theta}{\Sigma}\,\mathrm{d}\theta\mathrm{d}\varphi + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2$$
$$+ \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\varphi^2,$$

where

$$\Delta = r^2 - 2Mr + a^2\,, \qquad \Sigma = r^2 + a^2\cos^2\theta.$$

In the limit $M \rightarrow 0$, gravity disappears and General Relativity reduces to Special Relativity, therefore the Kerr spacetime must reduce to the Minkowski spacetime. Check that this is indeed the case.

References

1. L.D. Landau, E. Lifschitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1989)
2. R.M. Wald, *General Relativity* (Chicago University Press, Chicago, 1984)
3. S.M. Carroll, *Spacetime and Geometry, An Introduction to General Relativity* (Addison-Wesley, San Francisco, 2004)
4. R. d'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 2002)